Optimal recovery of derivative of an analytic function

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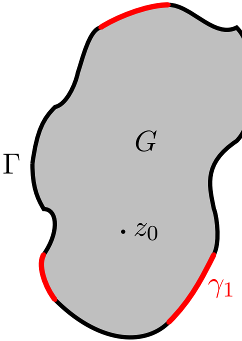

G — a simply connected bounded domain;

$\Gamma = \partial G$ — a closed rectifiable Jordan curve;

γ_1 — Lebesgue measurable subset Γ; $\mu(\gamma_1) > 0$;

$\gamma_0 = \Gamma \setminus \gamma_1$;

$z_0 \in G$.

Consider the Hardy space $H(G)$ of functions analytic and bounded on the domain G.

In the space $H(G)$, consider the class Q of functions satisfying the condition

$$\|f\|_{L^\infty(\gamma_0)} \leq 1.$$

Denote by $\Upsilon^1_{z_0}$ the functional which is defined on the subspace of $L^\infty(\gamma_1)$ formed by functions that are boundary values on γ_1 of functions from the space $H(G)$ and which assigns the value of derivative of a given analytic function at the point z_0 to the boundary values of the function on γ_1

$$\Upsilon^1_{z_0} f := f'(z_0).$$

The Modulus of Continuity of the Functional $\Upsilon_{z_0}^1$ on the Class Q

The function of variable $\delta \in [0, \infty)$ defined by the relation

$$\omega(\delta) := \sup \left\{ |f'(z_0)| \, : \, f \in Q, \, \|f\|_{L^\infty(\gamma_1)} \leq \delta \right\}, \tag{1}$$

is referred to as the modulus of continuity of the functional $\Upsilon_{z_0}^1$ on the class Q.

Along with the values of the quantity $\omega(\delta)$, an extremal function at which the upper bound is attained is also of interest.

It follows from the definition in (1) that, for functions from the space $H(G)$, the following sharp inequality holds:

$$|f'(z_0)| \leq \|f\|_{L^\infty(\gamma_0)} \, \omega \left(\frac{\|f\|_{L^\infty(\gamma_1)}}{\|f\|_{L^\infty(\gamma_0)}} \right).$$

The Problem of Optimal Recovery

For the set $\mathcal{R}$ of methods of recovery from which the optimal one is chosen we take the set $\mathcal{O}$ of all possible functionals or the set $\mathcal{L}$ of all linear functionals or the set $\mathcal{B}$ of all bounded functionals on $L^\infty(\gamma_1)$.

For a number $\delta > 0$ and a method of recovery $T \in \mathcal{R}$, define the value of the error of the method by the formula

$$\mathcal{U}(T,\delta) := \sup\left\{|f'(z_0) - Tq| \,:\, f \in Q,\, q \in L^\infty(\gamma_1),\, \|f-q\|_{L^\infty(\gamma_1)} \le \delta\right\}$$

Then,

$$\mathcal{E}_{\mathcal{R}}(\delta) := \inf\left\{\mathcal{U}(T,\delta) \,:\, T \in \mathcal{R}\right\} \tag{2}$$

is the quantity of optimal recovery of the value of derivative of an analytic function at the point z_0 (or, equivalently, of optimal recovery of the functional $\Upsilon^1_{z_0}$) by the methods of recovery $\mathcal{R}$ on functions of the class Q from their boundary values γ specified with the error δ.
The problem is to find the quantity $\mathcal{E}_{\mathcal{R}}(\delta)$ and to define an optimal way of the recovery, i.e., a functional at which the lower bound is attained.

The Problem of the Best Approximation of a Functional

Let $\mathcal{L}(N)$ be the set of bounded linear functionals on $L^\infty(\gamma_1)$ whose norm does not exceed the number $N > 0$. The quantity

$$U(T) := \sup\{|f'(z_0) - Tf| \,:\, f \in Q\}$$

is the deviation of a functional $T \in \mathcal{L}(N)$ from the functional $\Upsilon^1_{z_0}$ on the class Q. Correspondingly, the quantity

$$E(N) := \inf\{U(T) \,:\, T \in \mathcal{L}(N)\} \tag{3}$$

is the best approximation of the functional $\Upsilon^1_{z_0}$ by the set of bounded linear functionals $\mathcal{L}(N)$ on the class Q.
The problem is to calculate the quantity $E(N)$ and find an extremal functional at which the lower bound is attained.

The Relationship between Problems

As is known, the problem of optimal recovery of a linear functional on a convex centrally symmetric class of elements of a Banach space using the set $\mathcal{O}$ of all possible functionals admits an optimal linear bounded method, and the very quantity of the optimal recovery is equal to the modulus of continuity of the recovered functional. Hence

$$\omega(\delta) = \mathcal{E}_{\mathcal{O}}(\delta) = \mathcal{E}_{\mathcal{L}}(\delta) = \mathcal{E}_{\mathcal{B}}(\delta), \;\; \delta \geq 0. \tag{4}$$

Problem (3) is a special case of Stechkin's problem on the approximation of an unbounded operator by bounded linear operators on a class of elements of a Banach space. The problem of best approximation of an unbounded functional and the relationship between this problem and that of optimal recovery of the functional was studied in the most complete way. In particular, for problems (1) and (3), this relationship is expressed by the following relation:

$$E(N) = \sup_{\delta \geq 0} \left\{\omega(\delta) - N\delta\right\}, \;\; N > 0. \tag{5}$$

Let us denote by

$P(z,\zeta) := \frac{\partial G}{\partial \overline{n}}(z,\zeta)$, $\zeta \in \Gamma$, the density of harmonic measure of the domain G,
$G(z,\zeta)$ — the classical Green's function of the domain G,

w — the harmonic in the domain G function of a variable $z = x + iy$, whose value at the point is equal to the harmonic measure of the set γ_1 with respect to the point $z \in G$ and the domain G. For this function we have the representation

$$w(z) = w(z, \gamma_1, G) := \int_{\gamma_1} P(z,\zeta)\, |d\zeta|,$$

w is the solution of the Dirichlet problem

$$\begin{cases} \Delta w(z) = 0, \\ w|_\Gamma = \chi_{\gamma_1}, a.e. \end{cases}$$

For $\delta > 0$ we define the function $f_\delta \in Q$ by the equation

$$f_\delta(z) = h^\sigma(z),\ \sigma = \ln\delta, \quad h(z) = \exp\{w(z) + iv(z)\}, \tag{6}$$

where v is the harmonic function conjugate to w.

The function f_δ is analytic, bounded and it does not vanish in the domain G.
Almost everywhere on the boundary Γ of the domain G the following equation holds

$$|f_\delta(\zeta)| = \begin{cases} 1, & \zeta \in \gamma_0, \\ \delta, & \zeta \in \gamma_1. \end{cases}$$

We denote by $\kappa = \kappa(z_0)$, $\overline{\nu} = \overline{\nu}(z_0)$ and $t = t(z_0)$, respectively, the length, the direction and the argument of the gradient of the harmonic measure γ_1 with respect G at the point z_0, i.e. which are defined by the equations below

$$\kappa = \kappa(z_0) := |\overline{\nabla} w(z_0)|, \quad \overline{\nu} = \overline{\nu}(z_0) := \frac{\overline{\nabla} w(z_0)}{|\overline{\nabla} w(z_0)|}, \quad \overline{\nu} = (\cos t, \sin t).$$

$$\alpha = w(z_0), \quad \beta = 1 - \alpha = w(z_0, \gamma_0, G).$$

g is the function that specifies univalent map of a domain G onto a unit circle and satisfies the conditions

$$g(z_0) = 0, \quad g'(z_0) > 0.$$

$$\eta(z_0) := \frac{2\, g'(z_0)}{\kappa(z_0)}$$

In the case when $\delta \geq 0$ and $z_0 \in G$ satisfy the condition

$$|\ln \delta| < \eta(z_0), \tag{7}$$

define on the domain G the function F_δ by the equation

$$F_\delta(z) := \frac{g(z) - g_0}{1 - g(z)\overline{g_0}}\, f_\delta(z), \quad g_0 := -e^{-it}\frac{\kappa(z_0)|\ln \delta|}{2g'(z_0)} = -e^{-it}\frac{|\ln \delta|}{\eta(z_0)}.$$

It is clear that if the condition (7) holds then the function F_δ ia analytic in the domain G, the inequality

$$|F_\delta(z)| \leq |f_\delta(z)|, z \in G,$$

holds and the limit boundary values of the functions $|F_\delta|$ and $|f_\delta|$ coincide.

In the case when $\delta \geq 0$ and $z_0 \in G$ satisfy the condition

$$|\ln \delta| \geq \eta(z_0), \tag{8}$$

consider the functional T_δ^1 on $L^\infty(\gamma_1)$ defined by the equation

$$T_\delta^1 f := e^{-it(z_0)} \int_{\gamma_1} J_{z_0}(\zeta) \, \frac{f_\delta(z_0)}{f_\delta(\zeta)} \, f(\zeta) \, |d\zeta| \tag{9}$$

where

$$J_{z_0}(\zeta) := \frac{\partial P}{\partial \overline{\nu}}(z_0, \zeta) + \ln \delta \, \kappa(z_0) \, P(z_0, \zeta).$$

$$\Upsilon_{z_0}^0 f := f(z_0)$$

The extremal functions are the functions of the form cf_δ, $|c| = 1$;

an optimal method of recovery is the functional defined by the equation

$$T_\delta^0 f := \int_{\gamma_1} P(z_0, \zeta)\, \frac{f_\delta(z_0)}{f_\delta(\zeta)}\, f(\zeta)\, |d\zeta|.$$

$$T_\delta^1 f := e^{-it(z_0)}\, \frac{\partial}{\partial \overline{\nu}} T_\delta^0 f$$

$\delta = 1$

Separately, we single out the case $\delta = 1$. Then $\ln \delta = 0$ and a consequence of the Schwarz-Pick lemma is the equation

$$\omega(1) = g'(z_0).$$

$cF_1 = cg$, $|c| = 1$, are the extremal functions.

For this case we define the functional T_1^1 by the equation

$$T_1^1 f := \frac{g'(z_0)}{2\pi i} \int_{\gamma_1} \frac{g'(\zeta)}{g^2(\zeta)} f(\zeta)\, d\zeta = \int_{\gamma_1} P(z_0, \zeta) \frac{g'(z_0)}{g(\zeta)} f(\zeta)\, |d\zeta|. \quad (10)$$

Theorem 1

The following assertions hold for the quantities (1) and (2).

(I) In the case $|\ln\delta| \geq \eta(z_0)$ the following equality holds

$$\omega(\delta) = \mathcal{E}_{\mathcal{O}}(\delta) = \mathcal{E}_{\mathcal{B}}(\delta) = \mathcal{E}_{\mathcal{L}}(\delta) = \kappa(z_0)\delta^{\alpha}|\ln\delta|.$$

The extremal functions in (1) are the functions of the form $cf_\delta,\ |c| = 1$; in the problem (2) an optimal method of recovery is the functional T_δ^1, defined by the equation (9).

(II) In the case $|\ln\delta| < \eta(z_0)$ the following equality holds

$$\omega(\delta) = \mathcal{E}_{\mathcal{O}}(\delta) = \mathcal{E}_{\mathcal{B}}(\delta) = \mathcal{E}_{\mathcal{L}}(\delta) = \kappa(z_0)\delta^{\alpha}\ \frac{1}{2}\left(\eta(z_0) + \frac{\ln^2\delta}{\eta(z_0)}\right).$$

The extremal functions in (1) are the functions of the form $cF_\delta,\ |c| = 1$.

(III) When $\delta = 1$ the following equality holds

$$\omega(1) = \mathcal{E}_{\mathcal{O}}(1) = \mathcal{E}_{\mathcal{B}}(1) = \mathcal{E}_{\mathcal{L}}(1) = g'(z_0).$$

The extremal functions in (1) are the functions of the form cg, $|c| = 1$; in the problem (2) an optimal method of recovery is the functional T_1^1 defined by the equation (10).

Theorem 2

For the quantity (3) the following assertions hold.
(I^*) If $N > 0$ has the form

$$N = \kappa(z_0)\,\delta^{-\beta}\,|\alpha \ln \delta + 1|, \quad |\ln \delta| \geq \eta(z_0), \tag{11}$$

then for the quantity (3) the following equality holds

$$E(N) = \kappa(z_0)\,\delta^{\alpha}\,|\beta \ln \delta - 1|.$$

The functional T^1_δ defined by the formula (9) is the functional of the best approximation.
(II^*) If $N > 0$ has the form

$$N = \kappa(z_0)\,\delta^{-\beta}\,\left[\frac{\alpha}{2}\left(\eta(z_0) + \frac{\ln^2 \delta}{\eta(z_0)}\right) + \frac{\ln \delta}{\eta(z_0)}\right], \quad |\ln \delta| < \eta(z_0), \tag{12}$$

then for the quantity (3) the following equality holds

$$E(N) = \kappa(z_0)\,\delta^{\alpha}\,\left[\frac{\beta}{2}\left(\eta(z_0) + \frac{\ln^2 \delta}{\eta(z_0)}\right) - \frac{\ln \delta}{\eta(z_0)}\right].$$

(III^*) When $N = \alpha g'(z_0)$ for the quantity (3) the equality $E(N) = \beta g'(z_0)$ holds. The functional of the best approximation is the functional T^1_1, defined by the equality (10).