Approximation by Θ-means of Walsh-Fourier series

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Pécs

Walsh group

Let denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition.

$$G := \mathop{\times}_{k=0}^{\infty} \mathbb{Z}_2.$$

The group operation on G is the coordinate-wise addition (denoted by $+$). Let be every subset open on $\mathbb{Z}_2$ and the topology is the product one on G.

The elements of Walsh group G are sequences of numbers 0 and 1, that is $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$.

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Measure

The direct product μ of the measures

$$\mu_2(\{0\}) := \mu_2(\{1\}) := \frac{1}{2}$$

is the Haar measure on G with $\mu(G) = 1$.

General introduction

Dyadic intervals

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

for $x \in G$. (A base for the neighbourhoods of G.) $I_n := I_n(0)$.

L^p spaces

The usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

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The modulus of continuity in $L^p, 1 \le p \le \infty$, of a function $f \in L^p$

$$\omega_p(f,\delta) := \sup_{|t|<\delta} \|f(.+t) - f(.)\|_p, \quad \delta > 0,$$

with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G.$$

The Lipschitz classes in L^p

For each $\alpha > 0$ defined by

$$\mathrm{Lip}(\alpha, p) := \{f \in L^p : \omega_p(f,\delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

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Walsh-Fourier analysis

Rademacher functions

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

Each natural number n can be uniquely expressed in the form

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\} \ (i \in \mathbb{N}).$$

Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$.

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Walsh-Fourier analysis

Walsh-Paley functions (Walsh (1923), Paley (1932))

$w_0 := 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

$\mathcal{P}_n$, the collection of Walsh polynomials of order less than n

$$P(x) = \sum_{k=0}^{n-1} a_k w_k(x),$$

where $\{a_k\}$ is a sequence of real numbers ($n \geq 1$).

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Some of the usual definitions of the Fourier analysis.

Fourier coefficients

$$\hat{f}(n) := \int_G f(x) w_n(x) d\mu(x) .$$

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Dirichlet kernels

$$D_n := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbb{P}$, $D_0 := 0$.

$$D_{2^A+j}(x) = D_{2^A}(x) + r_A(x)D_j(x), \quad j = 0, 1, \ldots, 2^A - 1.$$

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$$K_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} D_i(x) \quad (n \in \mathbb{P})$$

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Fejér, (general) Nörlund and weighted means

$$\sigma_n(f;x) := \frac{1}{n}\sum_{k=1}^{n} S_k(f;x),$$

$$t_n(f;x) := \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k}S_k(f;x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k \ (n \geq 1), \ q_0 > 0, \ q_k \geq 0$ and $\lim_{n\to\infty} Q_n = \infty$,

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Historical notes

1. Móricz and Siddiqi (1992) studied the rate of the approximation by Nörlund means of Walsh-Fourier series of a function in L^p (in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$). They generalized some results of Yano (1951), Jastrebova (1966) and Skvortsov (1981).
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Θ-summation

Sequence of matrices Θ_n

$$\Theta_n := \begin{pmatrix} \theta_{0,1} & 0 & 0 & \dots & 0 \\ \theta_{0,2} & \theta_{1,2} & 0 & \dots & 0 \\ \theta_{0,3} & \theta_{1,3} & \theta_{2,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{0,n} & \theta_{1,n} & \theta_{2,n} & \dots & \theta_{n-1,n} \end{pmatrix}$$

We always suppose that $\theta_{0,k} = 1$ for all $k \in \{1, \dots, n\}$ (real elements).

Notation: $\Delta\theta_{k,n} := \theta_{k+1,n} - \theta_{k,n}$.

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Θ-means and kernels

$$\sigma_n^\Theta(f,x) := \sum_{k=0}^{n-1} \theta_{k,n} \hat{f}(k) w_k(x), \quad K_n^\Theta(x) := \sum_{k=0}^{n-1} \theta_{k,n} w_k(x)$$

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Motivation

It is easily seen that

$$\sigma_n^{\Theta}(f,x) = \int_G f(t) K_n^{\Theta}(t+x) d\mu(t),$$

and that partial sums ($\theta_{k,n} = 1$), Fejér ($\theta_{k,n} = \frac{n-k}{n}$), Nörlund ($\theta_{k,n} = \frac{\sum_{i=0}^{n-k-1} q_i}{Q_n}$) and the weighted means ($\theta_{k,n} = \frac{\sum_{i=k+1}^{n} p_i}{P_n}$) are special Θ-means ($0 \le k < n$).

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Results with respect to Θ-means

Theorem 1. ($f \in L^p$, $0 \le \theta_{k,n} \downarrow$, a: concave, b: convex)

Let $f \in L^p$, $(1 \le p \le \infty)$ and $2 < n \in \mathbb{N}$. Let the finite sequence $\{\theta_{k,n} : 0 \le k \le n-1\}$ of nonnegative numbers be nonincreasing.

a.) Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \le k \le n-1\}$ be nonincreasing. Moreover, we suppose that $\theta_{n-1,n} = O\left(\frac{1}{n}\right)$. Then

$$\|\sigma_n^{\Theta}(f) - f\|_p \le 5 \sum_{j=0}^{|n|-1} 2^j |\Delta\theta_{2^{j+1}-1,n}| \omega_p\left(f, 2^{-j}\right) + O(\omega_p(f, 2^{-|n|})).$$

b.) Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \le k \le n-1\}$ be nondecreasing. Then

$$\|\sigma_n^{\Theta}(f) - f\|_p \le 5 \sum_{j=0}^{|n|-1} 2^j |\Delta\theta_{2^j-1,n}| \omega_p\left(f, 2^{-j}\right) + O(\omega_p(f, 2^{-|n|})).$$

Results with respect to Θ-means

Theorem 2. ($f \in \mathrm{Lip}(\alpha, p)$, $0 \le \theta_{k,n} \downarrow$, a: concave, b: convex)

Let $f \in \mathrm{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \le p \le \infty$. For Θ-mean σ_n^Θ, we suppose the conditions in Theorem 1.
In case a.) the next estimate holds

$$\|\sigma_n^\Theta(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(\log n/n), & \text{if } \alpha = 1, \\ O(1/n), & \text{if } \alpha > 1. \end{cases}$$

In case b.) this holds

$$\|\sigma_n^\Theta(f) - f\|_p = O\left(\sum_{j=0}^{|n|-1} |\Delta\theta_{2^j-1,n}| 2^{j(1-\alpha)} + 2^{-|n|\alpha}\right).$$

Results with respect to Θ-means

Theorem 3.a. ($f \in L^p$, $\theta_{k,n} \downarrow$, concave, $\theta_{n-1,n} < 0$)

Let $f \in L^p$, $(1 \le p \le \infty)$ and $2 < n \in \mathbb{N}$. Let the finite sequence $\{\theta_{k,n} : 0 \le k \le n-1\}$ be nonincreasing and $\theta_{n-1,n} < 0$.

Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \le k \le n-2\}$ be nonincreasing. Moreover, we suppose that $|\theta_{n-1,n}| = O\left(\frac{1}{\sqrt{n}}\right)$ and $|\Delta\theta_{n-2,n}| = O\left(\frac{1}{n}\right)$. Then

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Theorem 3.b.($f \in L^p,\ \theta_{k,n} \downarrow,$ convex, $c_* \le \theta_{n-1,n} < 0$)

Let $f \in L^p$, $(1 \le p \le \infty)$ and $2 < n \in \mathbb{N}$. Let the finite sequence $\{\theta_{k,n} : 0 \le k \le n-1\}$ be nonincreasing and we suppose, that there exists a negative constant c_*, such that $c_* \le \theta_{n-1,n} < 0$ for all n.

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Theorem 4. $(f \in L^p,\ \theta_{k,n} \uparrow,$ concave, $\theta_{n-1,n} \le c^*)$

Let $f \in L^p$, $(1 \le p \le \infty)$ and $2 < n \in \mathbb{N}$. Let the finite sequence $\{\theta_{k,n} : 0 \le k \le n-1\}$ be nondecreasing. We suppose that there exists a positive constant c^* such that $\theta_{n-1,n} \le c^*$ for all n.

Let the finite sequence of differences $\{\Delta\theta_{k,n} : 0 \le k \le n-2\}$ be nonincreasing. Then

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Remarks

1. **Lemma (Móricz, Schipp (1990))**

   For every $1 < p \leq 2$, sequence $\{a_k\}$ of real numbers, and integer $n \geq 1$,
   $$\left\|\sum_{k=1}^{n} a_k D_k\right\|_1 \leq \frac{2p}{p-1}\left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p}.$$

   With $p := 2$ we used it for $\left\|\sum_{k=0}^{n-2^{|n|}} \Delta\theta_{2^{|n|}+k-1,n} D_k(x)\right\|_1 \leq c$, and it will not hold in case $\theta_{k,n} \uparrow$ and convex.

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Thank you for your attention!