The Dirichlet problem in weighted norm

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The problem

w - weight function

For any $f \in C(w)$ find a **harmonic** function u_f on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\lim_{r \to 1-} \|u_f(r, \theta) - f(\theta)\|_{C(w)} = 0,$$

where $z = re^{i\theta}$.

(Solution of the classical problem: convolution of f with the Poisson kernel $P_r(x)$.)

The weight

$$w(x) = v(x) \prod_{j=1}^{s} \left| \sin\left(\frac{x - x_j}{2}\right) \right|^{\lambda_j},$$

$v(x)$ is positive, continuous on $\mathbb{T}$, ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.)

$$\max_{(x,t)\in\mathbb{T}^2} \frac{v(x)}{v(t)} \le C_0, \ ;$$

$X = \{x_1, x_2, \ldots, x_s\} \subset \mathbb{T}$: distinct points, $\Lambda = \{\lambda_j\}_{j=1}^{s}$: positive real numbers.

$$C(w) := \{f : \mathbb{T} \to \mathbb{C} : fw \in C(\mathbb{T}),\ \lim_{x\to x_j} f(x)w(x) = 0, \quad j = 1, \ldots, s.,$$

$$\|f\|_{C(w)} = \max_{x\in\mathbb{T}} |f(x)|w(x).$$

The result

Theorem 1 *Let* $\Lambda \bigcap \mathbb{N} = \emptyset$ *For any* $f \in C(w)$ $\exists!$ u_f *on the unit disk s. t.* u_f *is harmonic,* $\lim_{r\to 1-} \|u_f(r,\theta) - f(\theta)\|_{C(w)} = 0$*, moreover*

$$u_f(r,\theta) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t) P_{X,\Lambda,r}(\theta,t)\, dt.$$

Theorem 2 *Let* $\Lambda \bigcap \mathbb{N} \neq \emptyset$*. Then there exists* $f \in C(w)$ *such that*

$$\limsup_{r\to 1-} \sup_{x\in\mathbb{T}} w(x) \int_{\mathbb{T}} f(t) P_{X,\Lambda,r}(x,t) dt = +\infty.$$

$P_{X,\Lambda,r}$

$k_j := [\lambda_j]$, $|\Lambda|_* := \sum_{j=1}^{s} k_j$, $T_{j,l}(x)$ trig. polynomials of degree n:

$$T_{j,l}^{(m)}(x_i) = \delta_{l,m}\delta_{i,j},$$

$1 \leq i, j \leq s$; $0 \leq m \leq k_i - 1$; $0 \leq l \leq k_j - 1$.

(If $|\Lambda|_* = 2n$, the quotient of the leading coefficients are given.)

If $|\Lambda|_* > 0$:

$$P_{X,\Lambda,r}(x,t) := P_r(t-x) - \sum_{j=1}^{s} \sum_{l=0}^{k_j-1} P_r^{(l)}(x_j - x) T_{j,l}(t)$$

and

$$P_{X,\Lambda,r}(x,t) := P_r(t-x) \quad \text{if} \quad |\Lambda|_* = 0.$$

The main step

Theorem 3 *For any weight function w, where w satisfies the conditions above and $\Lambda \bigcap \mathbb{N} = \emptyset$ there exists $C > 0$ such that*

$$\sup_{0<r<1} \sup_{x \in \mathbb{T}} w(x) \int_{\mathbb{T}} \frac{1}{w(t)} |P_{X,\Lambda,r}(x,t)| dt \leq C.$$

Theorem 3 $\rightarrow$ Theorem 1

A system of elements $\Phi = \{\varphi_n\}_{n=1}^{\infty} \subset B$ is an $A-$basis of the Banach space B if Φ is closed and minimal in B and for any $x \in B$

$$\lim_{r \longrightarrow 1-} \|x - \sum_{n=1}^{\infty} r^n \phi_n^*(x) \varphi_n\|_B = 0,$$

where Φ^* is the conjugate system of Φ (i.e. $\phi_n^*(\varphi_k) = \delta_{nk}\, n, k \in \mathbb{N}$).

Lemma 1 *Let* $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ *is a closed and minimal system in a separable Banach space* B. *Then* Φ *is an* $A-$*basis of* B *if and only if there exists a constant* $C > 0$ *such that for any* $x \in B$

$$\sup_{0<r<1} \| \sum_{n=1}^{\infty} r^n \phi_n^*(x) \varphi_n \|_B \leq C \|x\|_B.$$

Lemma 2 *Let* w *be a weight function, where* w *satisfies the conditions above. Then* $\tau \in C^*(w)$ *if and only if there exists a unique class of equivalences* $E_\tau \in C_{\mathbb{T}}^*/\mathcal{M}_X$ *of complex Borel measures such that*

$$\tau(f) = \int_{\mathbb{T}} f(t) w(t) d\mu(t) \qquad \forall f \in C(w) \quad \textit{and} \quad \forall \mu \in E_\tau,$$

and

$$\|\tau\|_{C^*(w)} = \|E_\tau\|_{C^*/\mathcal{M}_X}.$$

$\mathbb{Z}^*_\Lambda = \{k \in \mathbb{Z} : k = -n, n, -n-1, n+1, \dots\}$ if $|\Lambda|_* = 2n-1$,
($|\Lambda|_* = 2n$ similarly.)

$$\mathcal{T}_\Lambda = \{e^{ikx} : k \in \mathbb{Z}^*_\Lambda\}\, n = 1, \dots; \qquad \text{if} \quad |\Lambda|_* = 2n-1.$$

$$\mathcal{T}_\Lambda = \{a_n \cos nx + b_n \sin nx, e^{ikx} : k \in \mathbb{Z}^*_\Lambda\} \ \text{if}\ |\Lambda|_* = 2n,$$

Lemma 3 *Let w as above. Then the system $\mathcal{T}_\Lambda$ is closed and minimal in $C(w)$ with the conjugate system $\{E_k\}_{k\in\mathbb{Z}^*_\Lambda}$ if $|\Lambda|_* = 2n-1$ ($\{E_n, E_k\}_{k\in\mathbb{Z}^*_\Lambda}$, $|\Lambda|_* = 2n$,) where $E_k \in C^*/\mathcal{M}_X$. Moreover, $dg_k \in E_k$*

$$dg_n(x) = \frac{1}{\pi(a_n^2 + b_n^2)w(x)}\left(a_n \cos nx + b_n \sin nx - \sum_{j=1}^{s}\sum_{l=0}^{k_j-1}\frac{d^l}{dt^l}(a_n \cos nt + b_n \sin nt)\Big|_{t=x_j} T_{j,l}(x)\right)dx,$$

$$dg_k(x) = \frac{1}{w(x)2\pi}\left(e^{ikx} - \sum_{j=1}^{s}\sum_{l=0}^{k_j-1}\frac{d^l}{dt^l}e^{ikt}\Big|_{t=x_j} T_{j,l}(x)\right)dx.$$

Related results

1. Abel summation in Hermite-type w. s. with singularities (H. 2007)

The problem. v - weight on I, $\{\varphi_n\}$ - ONS on I w.r.t. v.
Poisson integral of a function f:

$$P_r(f,x) = \sum_n r^n a_n(f)\varphi_n, \ (a_n(f) \text{ - Fourier coefficients of } f)$$

Q:

? $1 \le p \le \infty$, $\lim_{r\to 1-} \|f - \sum_{n=M}^{\infty} r^n a_n(f) h_n\|_{v,p} = 0$?

$$I=\mathbb{R},\ v=ws,$$

$w(x)=e^{-\frac{x^2}{2}}$, s has zeros at $X:=\{x_1,x_2,\ldots,x_s\}$, e.g.
$s(x)=\prod_{j=1}^{s}|x-x_j|^{k_j+\delta_j}e^{-c|x|^\alpha}$, $\frac{-1}{p}<\delta_j<\frac{1}{p'}$, $\alpha<2$.
h_k - ON Hermite polynomials.

- $M:=\sum_{j=1}^{s}k_j$, then $H=\{h_k\}_{k=M}^{\infty}$ is complete and minimal in L^p_{ws}.

- $k\geq M$, $h_k^*=\frac{1}{s^2}\left(h_k-\sum_{l=0}^{M-1}a_{l,k}h_l\right)$ - the dual system.

- $P_s(r,x,y):=P_r(x,y)-\sum_{j=1}^{s}\sum_{l=0}^{k_j-1}\left.\frac{\partial^l P_r(x,y)}{\partial y^l}\right|_{y=x_j}H_{l,j}(y)$,

$\left(\left(H_{l,j}\right)^{(m)}(x_i)=\delta_{i,j}\delta_{l,m}\right)$

$$\sup_{0\leq r<1}\left\|\left(\int_{\mathbf{R}}f(y)P_s(r,x,y)e^{-y^2}dy\right)s(x)\right\|_{w,p}\leq c\|fs\|_{w,p};\ \ 1<p<\infty.$$

Theorem 4 *$\{h_k\}_{k=M}^{\infty}$ is A-basis in L^p_{ws}.*

2. Biorth. systems in Freud-type w. s. with inf. many zeros (H. 2011)

The problem. wv is a weight on $\mathbb{R}$.
- w is a Freud weight, e. g. $w(x) = e^{-|x|^\alpha}$, $\alpha > \frac{3}{2}$.
$X := \{x_1, x_2, \dots\} \subset \mathbb{R}$, $0 < |x_1| \leq |x_2| \leq \dots$; $M := \{m_1, m_2, \dots\} \subset \mathbb{R}_+$ $\varrho \geq 0$ s. t. $\sum_{j=1}^{\infty} \frac{m_j}{|x_j|^{\varrho+\varepsilon}} < \infty$, but $\sum_{j=1}^{\infty} \frac{m_j}{|x_j|^{\varrho-\varepsilon}} = \infty$ $\forall \varepsilon > 0$,
$\mu, d > 0$ arbitrary,

$$v(x) = v_{X,M,\mu,d}(x) := e^{d|x|^{\varrho+\mu}} \prod_{j=1}^{\infty} \left| 1 - \frac{x}{x_j} \right|^{m_j}.$$

- $\{\varphi_n\}$ - ONS on I w.r.t. w.

Q:

? $\{\varphi_n\} = \{\varphi_{n_k}\} \bigcup \{\varphi_{l_m}\}$; $\{\varphi_{n_k}\} \bigcap \{\varphi_{l_m}\} = \emptyset$ and $\{\varphi_{n_k}\}$ is complete and minimal in $L^p_{v_{X,M,\mu,d}}$?

Theorem 5 *w - Freud weight (+ assumtions...), $0 < \gamma$ (+a), g function (+a) M system of positive numbers (+a). Then there exists a system of points $X \subset \mathbb{R}$ and an "omission system" $\Psi_m = p_{l_m}(w)w$ with*

$$l_m = g(m) + O(m),$$

and $d, \mu > 0$, such that the system

$$\{\varphi_{n_k}\}_{k=1}^{\infty} := \{p_k(w)w\}_{k=0}^{\infty} \setminus \{\Psi_m\}_{m=1}^{\infty}$$

is complete and minimal in $L^p_{v_{X,M,\mu,d}}$, where $\inf_{m_j<1} \frac{1}{1-m_j} > p > \max_{\substack{j \\ \gamma-m_j<0}} \frac{1}{\gamma-m_j+1}$, if there are finite many m_j-s for which $\gamma - m_j < 0$, and for $\inf_j \frac{1}{1-m_j} > p > 1$, if $\gamma - m_j \geq 0$ for all j.

Example (+a): $w(x) = e^{-|x|^6}$ then $\gamma = \frac{1}{4}$, $g(x) = x^{16}$, $0 < m_j < 1 + \frac{1}{4}$, and $\liminf_{j\to\infty} m_j > 0$

The dual system:

$$\varphi^*_{n_k} = \frac{\varphi_{n_k} - \sum_{i=1}^{\infty} a_{i,n_k} \Psi_i}{v^2}.$$

q(1) - interpolation on infinitely many nodes - infinite linear equation system

q(2) - convergence of the sum

q(3) - numerical solution of q(1) - finite section method.