Steinhaus tiling sets

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Pecs 2017

Joint work with M. Papadimitrakis

The classical Steinhaus question

- Steinhaus (1950s): Are there $A, B \subseteq \mathbb{R}^2$ such that

$$|\tau A \cap B| = 1, \quad \text{for every rigid motion } \tau?$$

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- In tiling language:

$$\rho A \oplus B = \mathbb{R}^2, \quad \text{for all rotations } \rho.$$

Every rotation of A tiles (partitions) the plane when translated at the locations B.

Fixing $B = \mathbb{Z}^2$: the lattice Steinhaus question

- Can we have $\rho A \oplus \mathbb{Z}^2 = \mathbb{R}^2$ for all rotations ρ?
- Equivalent: A is a fundamental domain of all $\rho\mathbb{Z}^2$.
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- In higher dimension:
  K. & Wolff (1999), K. & Papadimitrakis (2002):
  $\Longrightarrow$ No measurable Steinhaus sets exist for $\mathbb{Z}^d$, $d \geq 3$.

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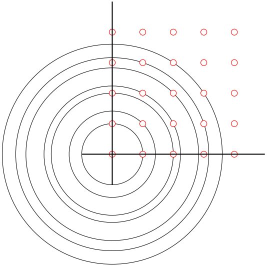
- Applying to $f = \mathbf{1}_{\rho A}$ for all rotations ρ we get



that $\widehat{\mathbf{1}_A}$ must vanish on all circles through lattice points.

The lattice Steinhaus question in Fourier space, cont'd

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- In dimension $d = 2$ this gives $\int_A |x|^{\frac{46}{27}+\epsilon} \, dx = \infty$.
- In dimension $d \geq 3$: better control of circle gap.
  We get $\mathbf{1}_A$ is continuous (contradiction)

The lattice Steinhaus question for finitely many lattices

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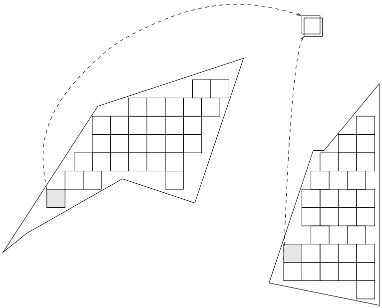
Generically yes!
If the sum $\Lambda_1^* + \cdots + \Lambda_n^*$ is direct then Kronecker-type density theorems allow us to rearrange a fundamental domain of one lattice to accomodate the others.



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- Fails in general: take $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the 3 copies of $\mathbb{Z}_2$ therein.
- No good condition is known!

An application in Gabor analysis

- **Question:** If K, L are two lattices in $\mathbb{R}^d$ with

$$\mathrm{vol}\, K \cdot \mathrm{vol}\, L = 1,$$

can we find $g \in L^2(\mathbb{R}^d)$, such that the (K, L) time-frequency translates

$$g(x-k)e^{2\pi i \ell \cdot x}, \quad (k \in K, \ell \in L)$$

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- The space is partitioned in copies of E and on each copy L is an orthogonal basis.

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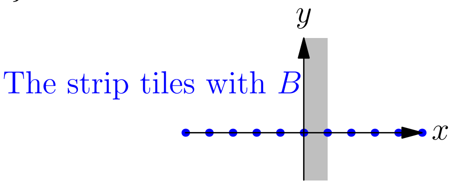

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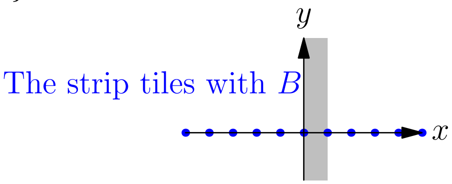

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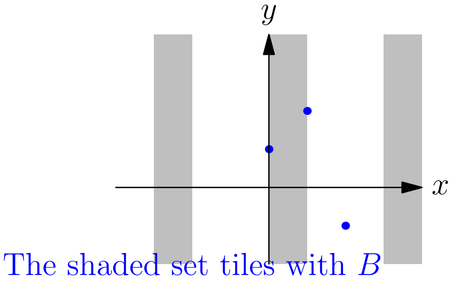

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WE SHOW HERE

- A Komjáth set cannot be Lebesgue measurable.
- For any finite $B \subseteq \mathbb{R}^2$ there is no Lebesgue measurable Steinhaus set A.

Finite B: a Fourier condition

Write $\delta_B = \sum_{b \in B} \delta_b$.

$\implies \widehat{\delta_B}(x) = \sum_{b \in B} e^{-2\pi i b \cdot x}$ is a trig. polynomial.

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$$\operatorname{supp} \widehat{\mathbf{1}_A} \subseteq \{0\} \cup \left\{\widehat{\delta_B} = 0\right\}.$$

(Notice $\widehat{\mathbf{1}_A}$ is a *tempered distribution*.)

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- Valid for all rotations ρ:

$$\bigcup_{\rho} \rho \left(\operatorname{supp} \widehat{\mathbf{1}_A} \right) \subseteq \{0\} \cup \left\{ \widehat{\delta_B} = 0 \right\}.$$

$\Longrightarrow$ The zeros of $\widehat{\delta_B}$ contain a *circle*.

Theorem

If $\psi(x) = \sum_{b \in B} c_b e^{2\pi i b \cdot x}$ *is a trigonometric polynomial on* $\mathbb{R}^d$ *which vanishes on a sphere then* $\psi(x) \equiv 0$.

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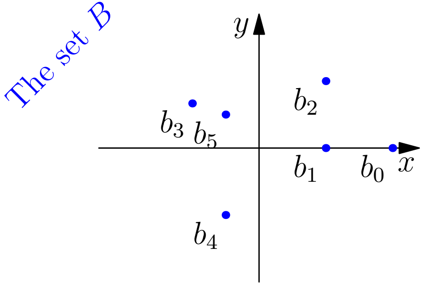

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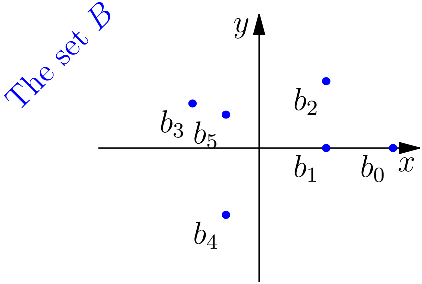

- Enough to prove for $d = 2$. May assume zeros at unit circle centered at origin.
- May also assume $(b_0, 0) \in B$ is unique with maximal modulus.

Zeros of trigonometric polynomials, continued

- Write $b = b_x + ib_y$, for $b \in B$, and $z = x - iy$, with $|z| = 1$. Then $(b_x, b_y) \cdot (x, y) = \Re(bz)$ and

$$\psi(x, y) \sum_{b \in B} c_b e^{2\pi i \Re(bz)} \stackrel{|z|=1}{=} \sum_{b \in B} c_b e^{\pi i (bz + \frac{\overline{b}}{z})} =: g(z)$$

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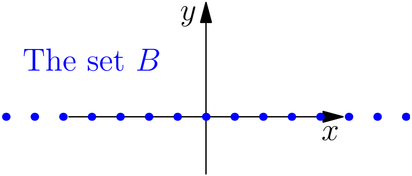
- For real $t \to +\infty$ we have

$$0 = g(-it) = c_{b_0} e^{\pi b_0 t + O(1/t)} + \sum_{b \in B \setminus \{(b_0, 0)\}} c_b e^{\pi i b t + O(1/t)}$$

  Contradiction for:
  unique exponential with highest exponent.

Komjáth sets



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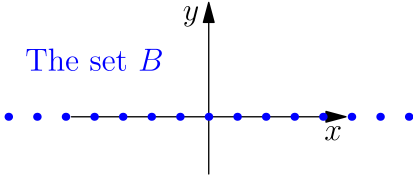

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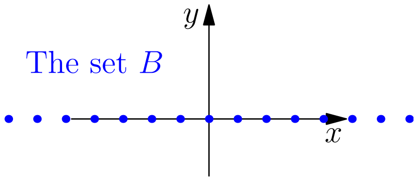

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- $\implies A$ has infinite measure.
- Integrating for $x \in [0,1]$ gives that

$$\rho A \cap \left(\mathbb{R} \times \{y\}\right) \quad \text{has measure 1 for almost all } y \in \mathbb{R}.$$

- Hence A intersects almost all lines of the plane at measure 1.

Komjáth sets: meeting the lines thus is too much

Theorem

There is no measurable $A \subseteq \mathbb{R}^2$ which intersects almost all lines of the plane in measure (length) at least C_1 and at most C_2, where $0 < C_1, C_2 < \infty$.

- We only need $C_1 = C_2 = 1$ for showing there are no measurable Komjáth sets.

Line integrals bounded above and below

- Suppose $A \subseteq \mathbb{R}^2$ has the bounded line intersection property. View $\mathbb{R}^2$ embedded in $\mathbb{R}^3$.
- Define $f : \mathbb{R}^3 \to \mathbb{R}^+$ by (convergence is clear)

$$f(z) = \int_{\mathbb{R}^2} \mathbf{1}_A(w) \frac{1}{|z - w|} \, dw.$$

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- **Claim:** $C_1\pi \leq f(z) \leq C_2\pi$ for almost all $z \in \mathbb{R}^2$

$$\begin{aligned} f(z) &= \int_{\mathbb{R}^2} \mathbf{1}_A(w) \frac{dw}{|z-w|} \\ &= \int_{\mathbb{R}^2} \mathbf{1}_A(z+w) \frac{dw}{|w|} \qquad \text{(change of variable)} \\ &= \int_{[0,\pi]} \int_{\mathbb{R}} \mathbf{1}_A(z + r(\cos\theta, \sin\theta))\, dr\, d\theta \qquad \text{(polar coordinates)} \\ &= \int_{[0,\pi]} |A \cap (z + L_\theta)|\, d\theta \quad \text{(where } L_\theta \text{ is the line with angle } \theta\text{)} \\ &\in [C_1\pi, C_2\pi]. \end{aligned}$$

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- Hence $C_1\pi \leq f(z) \leq C_2\pi$ <u>everywhere</u> on $\mathbb{R}^2$.
- f is harmonic in the upper half-space

$$H = \{(x_1, x_2, x_3) : x_3 > 0\}.$$

  Essentially because $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

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- If z' is the projection of $z \in \mathbb{R}^3$ onto $\mathbb{R}^2$ then

$$0 \leq f(z) \leq f(z') \leq C_2\pi.$$

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- If z' is the projection of $z \in \mathbb{R}^3$ onto $\mathbb{R}^2$ then

$$0 \le f(z) \le f(z') \le C_2\pi.$$

- Harmonic in H, bounded and continuous in $\overline{H} \implies$
  is the Poisson mean of $f \restriction \mathbb{R}^2 \implies$
  $C_1\pi \le f(z) \le C_2\pi$ for $z \in H$.

Line integrals bounded above and below, continued

- f is continuous on $\mathbb{R}^3$.
  Technical proof ommitted.
- Hence $C_1\pi \leq f(z) \leq C_2\pi$ everywhere on $\mathbb{R}^2$.
- f is harmonic in the upper half-space

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  Essentially because $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$.
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  is the Poisson mean of $f \restriction \mathbb{R}^2 \Longrightarrow$
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- Contradiction: Clearly $\lim_{t\to+\infty} f(x, y, t) = 0$.

The end.

Thank you.