Topics on Nörlund logarithmic means

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Content

- Almost everywhere convergence of some subsequences $(t_{m_n} f)_n$ of Nörlund logarithmic means of Walsh Fourier coefficients for every integrable function f and divergence for other classes of subsequences.
- Convergence and divergence in norm of Nörlund logarithmic means of generalized Walsh Fourier coefficients on some unbounded Vilenkin groups.

Motivation for Walsh-Paley system

- The Riesz logarithmic means of Walsh or trigonometric Fourier series $\frac{1}{\log n}\sum_{k=1}^{n-1}\frac{S_k f}{k}$ of any integrable function f converges almost everywhere to the original function f.
- This is not true for the Walsh Fourier series which diverges everywhere for some integrable function f satisfying $\int \varphi(|f|) < \infty$, where $\varphi(u) = o(u\sqrt{\log u})$.
- As Gát and Goginava mentioned the following results show a similarity of Nörlund logarithmic means with Walsh Fourier series rather than classical logarithmic means.

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Goginava's results for Walsh-Paley system

(Goginava 2005)

Theorem

Let $\{(m_n)_n : n \geq 1\}$ be a sequence of positive integers for which

$$\sum_{n=1}^{\infty} \frac{\log^2(m_n - 2^{\lfloor \log m_n \rfloor} + 1)}{\log m_n} < \infty.$$

Then, the operator $t^* f := \sup_{n\geq 1} |t_{m_n} f|$ is of weak type $(1,1)$.

Corollary

Let $\{(m_n)_n : n \geq 1\}$ be the sequence defined in the theorem above and f an integrable function. Then, $t_{m_n} f \to f$, a.e.

Goginava's results for Walsh-Paley system

Corollary

For all integrable function f, we have $t_{2^n} f \to f$, a.e.

- I. Blahota [1] proved the validity of the same results on the 2-adic group
- Bijection between 2-adic and dyadic integers
- Important difference between the two systems of characters

Motivation for Walsh-Paley system

(Gát-Goginava 2009):

Theorem

Let $\varphi : [0, \infty) \to [0, \infty)$ *be a function such that* $\frac{\varphi(u)}{u}$ *is nondecreasing and* $\varphi(u) = o(u\sqrt{\log u})$*. Then there exist a function* $f \in L$ *and a measurable set* E *with positive measure for which*

$$\int \varphi(|f|) < \infty$$

and

$$\limsup t_n f(x) = \infty, \forall x \in E.$$

Remark

Due to the corollary above it is impossible to replace $\limsup t_n f(x) = \infty$ *by* $\lim t_n f(x) = \infty$*, in this theorem.*

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Notations and definitions

The Nörlund logarithmic means are defined by

$$t_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \qquad l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

$$F_n := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k}{n-k},$$

$$t_n f = F_n * f.$$

Define the function $\varphi : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ by $\varphi(n) = n - 2^{[\log_2 n]}$.
Set $\varphi^1(n) = \varphi(n)$, $\varphi^0(n) = n$ and $\varphi^i(n) = \varphi \circ \varphi^{i-1}(n)$ when $i \geq 2$.

For every $n \in \mathbb{N} \setminus \{0\}$, $i \geq 0$, such that $\varphi^i(n) > 0$, define the functions $\alpha_i(n) = [\log_2(\varphi^i(n))]$ and $\beta_i(n) = l_{\varphi^i(n)}$.

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Main results

Theorem

Let $(m_n)_n$ be an increasing sequence of positive integers. Suppose that

$$\sum_{i:\varphi^i(m_n)>0} \frac{\beta_i(m_n)}{l_{m_n}} = O(1).$$

Then, $t_{m_n} f \to f$, a.e.

The condition of [4, Theorem 1] from which Goginava proves that $t_{m_n} f \to f$, a.e. formulated in our notations is

$$\sum_{n=1}^{\infty} \frac{\alpha_1^2(m_n)}{\alpha_0(m_n)} < +\infty.$$

Main results

Since $\sharp\{i : \varphi^i(n) > 0\} \leq \alpha_1(n)$, it follows that

$$\sum_{i:\varphi^i(m_n)>0} \frac{\alpha_i(m_n)}{\alpha_0(m_n)} < \frac{\alpha_1^2(m_n)}{\alpha_0(m_n)}.$$

If the sequence $(m_n)_n$ satisfies the condition of [4, Theorem 1], then

$$\alpha_1^2(m_n) = o(\alpha_0(m_n)),$$

which implies that

$$\sum_{i:\varphi^i(m_n)>0} \alpha_i(m_n) = o(\alpha_0(m_n)),$$

or equivalently,

$$\sum_{i:\varphi^i(m_n)>0} \beta_i(m_n) = o(\beta_0(m_n)) = o(l_{m_n}).$$

Therefore, this theorem is a generalization of [4, Theorem 1].

Main results

Theorem

Let $(m_n)_n$ and $(s_n)_n$ be increasing sequences of positive integers for which:

1. *the sequence $(\varphi^{s_n}(m_n))_n$ is increasing,*
2. $l_{m_n} = o(\beta_{s_n}(m_n)\sqrt{s_n})$, *when* $n \to \infty$,

then there exists an integrable function f such that $t_{m_n} f \nrightarrow f$ on a subset of positive measure.

Convergence in norm of logarithmic means for Walsh-Fourier coefficients

(F. Schipp, W.R. Wade, P. Simon, and J. Pál, 1990)

Theorem

Let $f \in C$ *and* $\omega(\delta, f)_\infty = o\left(\frac{1}{\log(\frac{1}{\delta})}\right)$, *then* $\| S_n f - f \|_\infty \to 0$.

(Gát-Goginava 2006)

Theorem

Let $f \in C$ *and* $\omega(\delta, f)_\infty = o\left(\frac{1}{\log(\frac{1}{\delta})}\right)$, *then* $\| t_n f - f \|_\infty \to 0$.

Theorem

There exists a function $g \in C$ *such that* $\omega(\delta, g)_\infty = O\left(\frac{1}{\log(\frac{1}{\delta})}\right)$, *and* $t_n g(0)$ *diverges.*

Convergence in norm of logarithmic means for Walsh-Fourier coefficients

(B.I. Golubov, A.V. Efimov, and V.A. Skvortsov, 1991)

Theorem

Let $f \in L^1$ *and* $\omega(\delta, f)_{L^1} = o\left(\frac{1}{\log(\frac{1}{\delta})}\right)$, *then* $\| S_n f - f \|_1 \to 0$.

(Gát-Goginava 2006)

Theorem

Let $f \in L^1$ *and* $\omega(\delta, f)_{L^1} = o\left(\frac{1}{\log(\frac{1}{\delta})}\right)$, *then* $\| t_n f - f \|_1 \to 0$.

Theorem

There exists a function $g \in L^1$ *such that* $\omega(\delta, g)_{L^1} = O\left(\frac{1}{\log(\frac{1}{\delta})}\right)$, *and* $\| t_n g - g \|_1 \nrightarrow 0$.

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Fourier series, Fejér means-Convergence in norm

- For Vilenkin systems $\| S_n f - f \|_p \to 0$, $1 < p < \infty$ (P. Simon 1976).
- $\| S_{M_n} f - f \|_1 \to 0$, (G. H. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli, A. I. Rubinstein, 1981).
- For bounded groups $\| \sigma_n f - f \|_p \to 0$, $1 \leq p < \infty$ (Simon, P., Pál, J., 1977).
- On arbitrary groups as a trivial consequence of the convergence of the partial sums: $\| \sigma_n f - f \|_p \to 0$, $1 < p < \infty$.
- On every unbounded group there exists an integrable function f such that $\| \sigma_{M_n} f - f \|_1 \nrightarrow 0$ (Price, J., 1957).
- For every Vilenkin system and every integrable function $\sigma_{M_n} f \to f$ a.e (Gát 2003).

Motivations for Part 2

- In general, the Fejér (C, 1) means have better properties, than the logarithmic ones. In the case of some unbounded Vilenkin systems the situation may change.
- In their paper [2] the authors have proved a convergence result of the subsequence $(t_{M_n} f)_n$ to the integrable function f in the L^1 norm for some unbounded Vilenkin groups.
- The main tool was the boundedness of the sequence $(\|F_{M_n}\|_1)_n$.
- Paradoxically, this is the reason for the divergence of the whole sequence $(t_n f)_n$.
- Therefore, in order to construct unbounded groups on which the sequence $(t_n f)_n$ converges in the L^1 norm, the property of uniform boundedness should be avoided.

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Blahota-Gát's results

Theorem

If $f \in L^p(1 \leq p < \infty)$ *and* $\limsup_n \|F_{M_n}\|_1 < \infty$, *then*

$$\| t_{M_n} f - f \|_p \to 0.$$

If f is continuous then the convergence holds in the supremum norm.

Theorem

If $\log m_n = O(n^\delta)$ *for some* $0 < \delta < \frac{1}{2}$, *then there exists an integrable function such that*

$$\| t_n f - f \|_1 \not\to 0.$$

Blahota-Gát's results

Example

Let

$$m_n = \begin{cases} \lfloor \exp(n^{\frac{1}{4}}) \rfloor, & \text{if } n = j^2,\ j \in \mathbb{N}; \\ 2, & \text{otherwise.} \end{cases}$$

In this case we have

1. $\limsup_n m_n = \infty$,
2. $\log m_n = O(n^{\frac{1}{4}})$,
3. $\limsup_n \|F_{M_n}\|_1 < \infty$.

Main result and examples

Theorem

If the sequence $(m_n)_n$ is unbounded and if the sequence $(F_{M_n})_n$ is bounded in L^1, then there exists a function $f \in L^1$ such that $t_n f \nrightarrow f$ in L^1.

Example

There exists an unbounded Vilenkin group represented by the sequence $(m_n)_n$ such that

1. $\log m_{n_k} \sim \sqrt{n_k}$, *for some subsequence $(m_{n_k})_k$ and*
2. $t_n f \nrightarrow f$ *in* L^1.

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Example

Using the the previous theorem and [2, Lemma 4] it suffices to construct a sequence $(m_n)_n$ such that

$$\sup_n \|F_{M_n}\|_1 \le \sup_n \frac{\sum_{k=0}^{n-1} (\log m_k)^2}{\sum_{k=0}^{n-1} \log m_k} < +\infty.$$

Let $m_k = 2$ if $k \neq 4^s$ for all positive integers s, and $\log m_k = 2^s = \sqrt{k}$ if $k = 4^s$. Hence we have

$$\sum_{k=0}^{n-1} (\log m_k)^2 = \sum_{s=[\log\sqrt{n-1}]+1}^{n-1} (\log 2)^2 + \sum_{s=0}^{[\log\sqrt{n-1}]} 4^s$$

$$\le n(\log 2)^2 + C4^{\log\sqrt{n}} \sim n,$$

$$\sum_{k=0}^{n-1} \log m_k \sim n \log 2 + 2^{\log\sqrt{n}} \sim n,$$

from which we easily obtain the result.

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