On a potential theoretic minimax problem on the torus

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We use the unit circle $\mathbb{T}$ which we usually identify with $[0, 2\pi)$. We take (possibly different) kernels K_j on $\mathbb{T}$ such that K_j's are convex on $(0, 2\pi)$, on considered on $\mathbf{R}$ they are 2π periodic, and K_j's have an infinite cusp at 0 meaning that

$$(\infty) \qquad \lim_{t \searrow 0} K_j(t) = -\infty = \lim_{t \nearrow 2\pi} K_j(t).$$

For a node system $\mathbf{y} = (y_0, y_1, \ldots, y_n)$,
$0 = y_0 < y_1 < \ldots < y_n < 2\pi$ consider the sum of translates function ("potential")

$$F(\mathbf{y}; t) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j).$$

Put $I_j = I_j(\mathbf{y}) := [y_j, y_{j+1}]$, $j = 0, 1, \ldots, n-1$, $I_n := [y_n, 2\pi]$ and investigate $m_j = m_j(\mathbf{y}) := \max\{F(\mathbf{y}; t) : t \in I_j(\mathbf{y})\}$ and $\mathbf{m} := (m_0, m_1, \ldots, m_n)$.

For example, take $K_j(t) = \log|t|$, $-\pi \le t \le \pi$, and then $F(\mathbf{y}, t) = \log|P(t)|$ where $P(t) = t(t - y_1)\dots(t - y_n)$ is a polynomial.

Or: if $\mathbf{w} : 0 \le w_1 \le w_2 \le \dots \le w_{2n} \le 2\pi$ is such that $w_j = 2\pi - w_{2n+1-j}$, $j = 1, 2, \dots, n$ and $K_j(t) = K(t) = \log|\sin(t/2)|$, then $F(\mathbf{w}; t) = \log|T(t)|$ where $T(t)$ is a trigonometric polynomial (with real coefficients) because

$$\sin\frac{t-\alpha}{2}\sin\frac{t+\alpha-2\pi}{2} = \frac{1}{2}(\cos t - \cos\alpha).$$

In the cases above, zeros are simple. If we want to consider zeros with higher but fixed multiplicity, take e.g. $K_j(t) = m_j \log|t|$ where $m_j \in \mathbb{N}$ is the multiplicity of y_j.

The problem of Bojanov

Consider polynomials with prescribed multiplicities of zeros in given order and minimize the sup norm over a fixed interval. How do(es) the extremal polynomial(s) behave?

Theorem (Bojanov, 1979 JAT)

Assume that a sequence of natural numbers $\nu_1, \nu_2, \ldots, \nu_k$ is given such that $\nu_1 + \ldots + \nu_k = n$ and $[a, b]$ is fixed. Then there is a unique polynomial (of degree n with leading coefficient 1 and) with minimal sup norm over $[a, b]$ (and with zeroes at $a < x_1 < \ldots < x_k < b$ with the corresponding multiplicities $\nu_1, \ldots, \nu_k$), and it is exactly the unique polynomial of degree n with the local maxima equioscillating (all equal).

This was established by Bojanov on $[-1, 1]$ (not on $\mathbb{T}$), by heavy classical type arguments (oscillation of polynomials, zero counting, monotonicity properties in function of the monotonic perturbation of nodes etc.).

(Cont.d)
A weight is not allowed here.

The analogous trigonometrical polynomial case has not been described ever since.

Some properties of potentials

An arbitrary sum of translates function (potential)

$$F(\mathbf{y}; t) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j)$$

where we assume that $0 = y_0 < y_1 < y_2 < \ldots < y_n < 2\pi$ has the following properties.

$F(\mathbf{y}; .)$ is concave on the interior (y_j, y_{j+1}) of each I_j.

If K_j is strictly concave, then each of the local maxima m_j is attained at a unique point in the interior of I_j.

$F(\mathbf{y}; t)$ is continuous in $\mathbf{y}$ and t when $0 = y_0 < y_1 < \ldots < y_n < 2\pi$ and $t \neq y_j$, $j = 0, 1, \ldots, n$ (we assume condition (∞)).

The function $\arctan F(\mathbf{y}; t)$ is continuous for any $y_1, y_2, \ldots, y_n, t \in \mathbb{T}$ (permitting $-\pi/2$ as value), and we say that $F(.; .)$ is continuous in the extended sense.

Some properties of local maxima

The maximum value $m_j(\mathbf{y})$ is continuous when $0 = y_0 < y_1 < \ldots < y_n < 2\pi$.

Proposition

Suppose that all the K_j *satisfy* (∞). *If* $\mathbf{y}^{(k)}$ *is a sequence of node configurations such that* $\min_{j=0,1,\ldots,n-1} \left| y_{j+1}^{(k)} - y_j^{(k)} \right| \to 0$ *as* $k \to \infty$ *then*

$$\lim_{k\to\infty} \max_{j=0,\ldots,n-1} \left| m_j(\mathbf{y}^{(k)}) - m_{j+1}(\mathbf{y}^{(k)}) \right| = \infty. \tag{1}$$

Proposition

If all K_j are C^2 smooth and $K_j'' < 0$ on $(0, 2\pi)$, then the functions $m_0, \dots, m_n$ are C^1 smooth.

Under the conditions of last Proposition, we introduce two properties:

Jacobi Property: The functions $m_0, \dots, m_n$ satisfy

$$\det\left(\partial_i m_j\right)_{i=1, j=0, j\neq k}^{n,n} \neq 0 \text{ for each } k \in \{0, \dots, n\}.$$

Difference Jacobi Property: The functions $m_0, \dots, m_n$ satisfy

$$\det\left(\partial_i (m_j - m_{j+1})\right)_{i=1, j=0}^{n,n-1} \neq 0.$$

Shi proved that under the condition (1) (see (∞) now) the Jacobi Property implies the Difference Jacobi Property.

Suppose that $K_0, \ldots, K_n$ are strictly concave. Let $\mathbf{y} \in \mathbb{T}^n$ be a node system, and for $k \in \mathbb{N}$, $1 \le k \le n$ let $t_1, \ldots, t_k \in (0, 2\pi)$ be all different from the nodes in $\mathbf{y}$. Let

$$\delta := \frac{1}{2} \min\left\{ |t_i - y_j| : i = 1, \ldots, k,\ j = 0, \ldots, n \right\}.$$

For $i = 1, \ldots, k$ let $\mu^{(i)}$ be the slope of a supporting line to the graph of $F(\mathbf{y}, \cdot)$ at the point t_i. Finally, let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-k} \in \mathbf{R}^n$ be fixed arbitrarily.

a) Then there is $\mathbf{a} \in [-1, 1]^n \setminus \{\mathbf{0}\}$ such that $\mathbf{x}_\ell^\top \mathbf{a} = 0$ for $\ell = 1, \ldots, n-k$ and for all $0 < h < \delta$ we have

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, t_i) + \mu^{(i)}(s_i - t_i)$$

for all s_i with $|s_i - t_i| < \delta$, $i = 1, \ldots, k$.

b) Let $S = S_\sigma$ be a simplex, and let $\mathbf{y} \in \overline{S}$. If $F(\mathbf{y}, \cdot)$ has local maximum in t_i for some $i \in \{1, \ldots, k\}$, i.e., if $t_i = z_j(\mathbf{y}) \in \operatorname{int} I_j(\mathbf{y})$ for some $j \in \{0, \ldots, n\}$, then

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, z_j(\mathbf{y})) = m_j(\mathbf{y})$$

for all s_i with $|s_i - z_j(\mathbf{y})| < \delta$.

Proposition

Suppose that for each $j = 0, \ldots, n$ the kernel K_j is C^2 with $K_j'' < 0$. The Jacobian matrix of

$$\Delta(\mathbf{y}) = (m_1(\mathbf{y}) - m_0(\mathbf{y}), \ldots, m_n(\mathbf{y}) - m_{n-1}(\mathbf{y}))^\top$$

is nonsingular where $0 = y_0 < y_1 < \ldots < y_n < 2\pi$. That is, we have the Difference Jacobi Property.

Actually, $-\Delta(\mathbf{y})$ has the following structure: it is positive on the diagonal, negative off the diagonal and, fortunately, the column sums are positive (hence the matrix is nonsingular).
Actually it is a P-matrix: each principal minor is positive, i.e. $A \in \mathbf{R}^{n \times n}, A = (a_{i,j})$, $\forall \alpha \subset \{1, \ldots, n\}$ with $A[\alpha] = (a_{i,j})_{i,j \in \alpha}$ we have $\det A[\alpha] > 0$.

There are several equivalent properties, and similar classes of matrices.

Corollary

Suppose that for each $j = 0, \ldots, n$ *the kernel* K_j *is* C^2 *with* $K_j'' < 0$ *and satisfies* (∞). *Then* $\Delta : \{0 = y_0 < y_1 < \ldots < y_n < 2\pi\} \to \mathbf{R}^n$ *is a homeomorphism.*

Corollary

Suppose that for each $j = 0, \ldots, n$ *the kernel* K_j *is* C^2 *with* $K_j'' < 0$ *and satisfies* (∞). *Then there is a unique equioscillation point.*

Main theorem in general setting

Thm: Suppose the kernel functions $K_0, K_1, \ldots, K_n$ are strictly concave and all belong to $\mathrm{C}^1(0,2\pi)$ (or all satisfy condition (∞')). Then there is $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with

$$M := \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y};t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*;t).$$

Moreover, we have the following:

a) $\mathbf{w}^*$ is an equioscillation point, i.e., $m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*)$.

b) $\mathbf{w}^* \in S$ for some simplex S, i.e., the nodes in $\mathbf{w}^*$ are different, and

$$\inf_{\mathbf{y}\in S} \max_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y};t) = M = \sup_{\mathbf{y}\in S} \min_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y};t).$$

c) We have the Sandwich Property on S, i.e., for each $\mathbf{x}, \mathbf{y} \in S$

$$\min_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{x})} F(\mathbf{x};t) \le M \le \max_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y};t).$$

Note that this theorem allows that the orderings of the nodes change (i.e. $\mathbf{w}^*$ can be in any simplex).

The previous theorem contains a global result: it finds extremal configuration $\mathbf{w}^*$ among all configurations $M = \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y}; t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*; t)$.
In general, extremal configuration within a simplex can be on the boundary of the simplex. In some cases, it is important to find extremal configuration under the given ordering (in the given simplex).

Lemma

Let K be strictly concave and let $a, b > 0$, $x, y \in (0, 2\pi)$ with $x \le y$ be given. Then for sufficiently small $\delta > 0$ we have that

$$\frac{1}{a}K(t-(y+ah)) + \frac{1}{b}K(t-(x-bh)) < \frac{1}{a}K(t-y) + \frac{1}{b}K(t-x)$$

for each $t \in (0, x - b\delta) \cup (y + a\delta, 2\pi)$ and each $0 < h < \delta$.

An important corollary of the main theorem

Suppose the kernel function K is strictly concave and either satisfy (∞') or is C^1. Let $r_0, r_1, \ldots, r_n > 0$, set $K_j := r_j K$ and

$$F(\mathbf{y}; t) := K_0(t) + \sum_{j=1}^{n} K_j(t - y_j) = r_0 K(t) + \sum_{j=1}^{n} r_j K(t - y_j).$$

Fix $S = \{0 = y_0 < y_1 < \ldots < y_n < 2\pi\}$.
Then there is a unique $\mathbf{w}^* \in S$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with $M(S) := \inf_{\mathbf{y} \in S} \sup_{t \in \mathbb{T}} F(\mathbf{y}; t) = \sup_{t \in \mathbb{T}} F(\mathbf{w}^*; t)$. Moreover, we have the following:

a) The nodes $w_0, \ldots, w_n$ are different and $\mathbf{w}^*$ is an equioscillation point, i.e.,

$$m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*).$$

b)

$$\inf_{\mathbf{y}\in S}\max_{j=0,\dots,n}\sup_{t\in I_j(\mathbf{y})}F(\mathbf{y};t)=M(S)$$
$$=m(S)=\sup_{\mathbf{y}\in S}\min_{j=0,\dots,n}\sup_{t\in I_j(\mathbf{y})}F(\mathbf{y};t).$$

c) We have the Sandwich Property on $\mathbb{T}$ in $\overline{S}$, i.e., for each $\mathbf{x},\mathbf{y}\in\overline{S}$

$$\min_{j=0,\dots,n}\sup_{t\in I_j(\mathbf{x})}F(\mathbf{x};t)\le M(S)\le\max_{j=0,\dots,n}\sup_{t\in I_j(\mathbf{y})}F(\mathbf{y};t).$$

General framework: trigonometric polynomials

Any trigonometric polynomial

$$T(t) = a_0 + \sum_{k=1}^{m} a_k \cos(kt) + b_k \sin(kt), \quad |a_m| + |b_m| > 0$$

can be written in the form

$$T(t) = c \prod_{j=1}^{2m} \sin \frac{t - x_j}{2}$$

with suitable $c, x_1, \ldots, x_{2m}$. Functions of the form

$$a \prod_{j=1}^{m} \left| \sin \frac{t - z_j}{2} \right|^{r_j} \text{ where } a, r_j > 0, \; z_j \in \mathbf{C}$$

is sometimes called generalized trigonometric polynomials (GTP).

Hence, put

$$K(x) := \log|\sin(x/2)|, \quad -\pi \le x \le \pi$$

and extend 2π periodically to $\mathbf{R}$. Then K is a C^2 smooth kernel with $K'' < 0$ (strictly concave) and satisfies (∞) (infinite cusp at 0).

Let $r_0, r_1, \ldots, r_n > 0$ be fixed. Then, there exists a unique system of points $\mathbf{w}^* = (w_0, w_1, \ldots, w_n)$, $0 = w_0 < w_1 < \ldots < w_n < 2\pi$ such that

$$\left\| \left|\sin\frac{t-w_0}{2}\right|^{r_0} \cdots \left|\sin\frac{t-w_n}{2}\right|^{r_n} \right\| = \inf_{0=y_0\le y_1<\ldots<y_n<2\pi} \left\| \left|\sin\frac{t-y_0}{2}\right|^{r_0} \cdots \left|\sin\frac{t-y_n}{2}\right|^{r_n} \right\|$$

where $\|\cdot\|$ denotes the sup-norm over $[0, 2\pi]$. The extremal GTP

$$T^*(t) := \left|\sin\frac{t-w_0}{2}\right|^{r_0} \cdots \left|\sin\frac{t-w_n}{2}\right|^{r_n}$$

has the properties that there exists $0 < z_0 < z_1 < z_2 < \ldots < z_n < 2\pi$ such that w_j's and z_j's interlace, i.e., $0 = w_0 < z_0 < w_1 < \ldots < w_n < z_n < w_0 + 2\pi = 2\pi$, and $T^*(z_j) = \|T^*\|$ for $j = 0, 1, \ldots, n$.

General framework in algebraic case

Fix $r_1, r_2, \ldots, r_n > 0$, and consider

$$|x - x_1|^{r_1} \ldots |x - x_n|^{r_n}$$

where such functions are sometimes called generalized algebraic polynomials (GAP). Consider the algebraic problem

$$(*) \qquad \inf_{a \le x_1 < \ldots < x_n \le b} \left\| |x - x_1|^{r_1} \ldots |x - x_n|^{r_n} \right\|$$

where the sup norm is over $[a, b]$ and the corresponding “doubled” trigonometric problem
(**)

$$\inf \left\| \left|\sin \frac{t - y_1}{2}\right|^{r_n} \ldots \left|\sin \frac{t - y_n}{2}\right|^{r_1} \left|\sin \frac{t - y_{n+1}}{2}\right|^{r_1} \ldots \left|\sin \frac{t - y_{2n}}{2}\right|^{r_n} \right\|$$

where the inf is taken for
$0 \le y_1 < \ldots < y_n < y_{n+1} < \ldots < y_{2n} < 2\pi$ and the sup norm is over $[0, 2\pi]$.

Theorem

Using the notations introduced above, (**) has unique solution $\mathbf{w}^* = (w_1, w_2, \ldots, w_{2n})$ *with* $w_1 + (w_{2n} - 2\pi) = 0$ *and* $0 < w_1 < \ldots < w_{2n} < 2\pi$. *Further,* $\mathbf{w}^*$ *is symmetric:* $w_k = 2\pi - w_{2n+1-k}$ *for* $k = 1, 2, \ldots, n$.

A more general symmetry theorem is the following:

Let $K_1, \ldots, K_n$ be strictly concave kernels such that K_j is even: $K_j(t) = K_j(-t)$ for all $j = 1, \ldots, n$. Assume that the kernels are in $\mathrm{C}^1(0, 2\pi)$. Take the simplex $S := \{0 \le y_1 < y_2 < \ldots < y_{2n} < 2\pi\}$. Define the symmetric potential

$$F_{symm}(\mathbf{y}, t) := K_1(t - y_1) + \ldots + K_{n-1}(t - y_{n-1}) + K_n(t - y_n) + K_n(t - y_{n+1}) + K_{n-1}(t - y_{n+2}) + \ldots + K_1(t - y_{2n})$$

and consider the "doubled" problem

$$M_{symm} := \inf_{\mathbf{y} \in S} \sup_{t \in [0, 2\pi)} F_{symm}(\mathbf{y}, t).$$

Then there is a unique solution $\mathbf{w}^* = (w_1, w_2, \ldots, w_{2n}) \in S$ with $w_1 + (w_{2n} - 2\pi) = 0$. Further, $\mathbf{w}^*$ is symmetric: $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \ldots, n)$ and there are exactly $2n$ points: $0 = z_1 < z_2 < \ldots < z_{n+1} = \pi < \ldots < z_{2n}$ where $F_{symm}(\mathbf{w}^*, .)$ attains its supremum. Moreover, z_j's and w_j's interlace and z_j's are symmetric too: $z_k = 2\pi - z_{2n+1-k}$.

Remark. In view of the rotation invariance on the torus, we can choose a normalization. Before we used $y_0 = 0$, now we prefer $w_1 = 2\pi - w_{2n}$.

Generalizing Bojanov's result

We connect the interval (algebraic) problem (*) and the trigonometric problem (**) using a classical idea transferring with $x = \cos(t)$ (e.g. Szegő 1964).

Theorem

Consider the algebraic problem (*) and the associated "doubled" trigonometric problem (**). Denote the unique solution of (**) by $\mathbf{w}^* = (w_1, \dots, w_{2n})$ *and that of (*) by* $\mathbf{x} = (x_1, \dots, x_n)$*, and let* $L(x) := \frac{b-a}{2}x + \frac{b+a}{2}$.
Then we can obtain $\mathbf{x}$ *from* $\mathbf{w}^*$: $x_j = L(\cos w_{n+1-j})$, $j = 1, \dots, n$.

From these the following generalization of Bojanov's result can be deduced immediately:

Theorem

Let $\nu_1, \dots, \nu_n > 0$ be fixed, and let $[a, b] \subset \mathbf{R}$. Then, there exists a unique system of points $a < x_1 < \dots < x_n < b$ such that

$$\| |x - x_1|^{\nu_1} \dots |x - x_n|^{\nu_n} \| = \inf_{a \le y_1 < \dots < y_n \le b} \| |x - y_1|^{\nu_1} \dots |x - y_n|^{\nu_n} \|$$

where $\| \cdot \|$ denotes the sup norm over $[a, b]$. The extremal GAP

$$P^*(x) := |x - x_1|^{\nu_1} \dots |x - x_n|^{\nu_n}$$

is uniquely characterized by the existence of $a = s_0 < s_1 < \dots < s_{n-1} < s_n = b$ with $P^*(s_j) = \|P^*\|$ for $j = 0, 1, \dots, n$.

Thank you for your attention!