Slice regular Malmquist-Takenaka systems

Margit Pap
University of Pécs, Hungary
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papm@gamma.ttk.pte.hu

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Summary

- In this talk I present the **slice regular analogue of the Malmquist-Takenaka system** in the quaternionic slice regular Hardy space. It is proved that, **under certain restrictions regarding to the parameters of the system, they form a complete orthonormal system** in the quaternionic Hardy spaces of the unit ball.
- The properties of the associated projection operator are studied.

The complex Malmquist-Takenaka system

- The first mention of rational orthonormal systems in the Hardy space of complex variable functions seems to have occurred in the work of **F. Malmquist (1925), S. Takenaka (1925)**.
- These systems can be viewed as **extensions of the trigonometric system** on the unit circle, that corresponds to the special choice **when all of the poles are located at the origin**.
- In the system theory they **are often used to identify the transfer function of the system**.

The complex Malmquist-Takenaka system

This orthonormal system is generated by a sequence $a = (a_1, a_2, ...)$ of complex numbers, $a_n \in \mathbb{D}$ of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke-functions

$$B_b(z) := \frac{z - b}{1 - \overline{b}z} \qquad (b \in \mathbb{D},\ z \in \mathbb{C}).$$

The **Malmquist-Takenaka system (M-T)** $\Phi_n = \Phi_n^a\,(n \in \mathbb{N}^*)$ is defined by

$$\Phi_1(z) = \frac{\sqrt{1 - |a_1|^2}}{1 - \overline{a_1}z},$$

$$\Phi_n(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n}z} \prod_{k=1}^{n-1} B_{a_k}(z),\ \ n \geq 2.$$

If $a_1 = a_2 = \ldots = 0$, then we reobtain the trigonometric system.

The complex Malmquist-Takenaka system

These functions form an orthonormal system on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, i.e.,

$$\langle \Phi_n, \Phi_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{it}) \overline{\Phi_m(e^{it})} dt = \delta_{mn} \quad (m, n \in \mathbb{N}^*),$$

where δ_{nm} is the Kronecker symbol. If the sequence $a = (a_1, a_2, ...)$ satisfies the **non-Blaschke condition**

$$\sum_{n \geq 1} (1 - |a_n|) = +\infty,$$

then the corresponding **M-T system is complete in the Hardy space of the unit disc**.

The complex Malmquist-Takenaka system

M. Pap, F. Schipp, 2001, 2003, 2004, 2015

- **the discrete orthogonality** property of these functions was proved,
- based on this a **quadrature method** and
- **interpolation formula** were introduced and studied
- The nodes of the discretization satisfy **equilibrium conditions for some potential functions**.

Analytic wavelets in the Hardy space of the unit disc

- **Problem of Yves F. Meyer (Abel Prize 2017)** Construction of analytic affine wavelets (using a mother wavelet, translations and dilations)
- **M. Pap, 2011 analytic hyperbolic wavelets**: for the Hardy space of the unit disc. Instead of dilations and translations (which appear in the definition of the representation of the affine group) we use a representation of the Blaschke group on to describe the multuresolution.
- **M. Pap, H. Feichtinger 2013** extension for the Hardy space of upper half plane.

Analytic hyperbolic wavelets in the Hardy space of the unit disc

Definition, M. Pap, JFAA, 2011

Let $V_j,\ j \in N$ be a sequence of subspaces of $H^2(\mathbb{T})$. The collections of spaces $\{V_j,\ j \in N\}$ is called a multiresolution if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$,

2. (density) $\overline{\cup V_j} = H^2(\mathbb{T})$

3. (analog of dilatation) $U_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$

4. (basis) There exist $\psi_{j\ell}$ (orthonormal) bases in V_j. Analytic hyperbolic wavelets: The Malmquist-Takenaka system with special localization of poles:

$$\psi_{m,\ell}(z) = \frac{\sqrt{1-r_m^2}}{1-\overline{z_{m\ell}}z} \prod_{k=0}^{m-1} \prod_{j=0}^{2^{2k}-1} \frac{z-z_{kj}}{1-\overline{z_{kj}}z} \prod_{j'=0}^{\ell-1} \frac{z-z_{mj'}}{1-\overline{z_{mj'}}z}.$$

The voice transform of the Blaschke group on $H^2(\mathbb{T})$

- **The representation of the Blaschke group on** $H^2(\mathbb{T})$: for $\left(z = e^{it} \in \mathbb{T}, a = (b, e^{i\theta}) \in \mathbb{B}\right), f \in H^2(\mathbb{T})$.
$(U_{a^{-1}} f)(z) := \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\overline{b}z)} f\left(\frac{e^{i\theta}(z-b)}{1-\overline{b}z}\right)$ **Not integrable, not square integrable.**
- The voice transform generated by U_a $(a \in \mathbb{B})$ is the **hyperbolic wavelet transform** given by the following formula
$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \quad (f, \rho \in H^2(\mathbb{T}))$.
- **Pap M., Schipp F.**, 2006, 2008, 2010, 2011,...
- The **matrix elements** of the representation can be given by the **Zernike functions** which play an important role in expressing the wavefront data in optical tests.
- An important consequence of this connection is the **addition formula for Zernike functions**

Problem: Extension for quaternions

- **Motivation:** Quaternions play an important role in modeling the time and space dependent problems in physics and engineering.
- Adler L. Stephen in Quaternionic quantum field theory provides an introduction to the problem of formulating quantum field theories in quaternionic Hilbert space. But the full power of quaternions would be even more important by using the quaternionic analysis.
- **Pap M., Schipp F.** (2004) and **Qian T., Sprossig W., Wang J.** (2012) respectively, following two different ways, introduced two analogues of the M-T systems in the set of quaternions. The drawback of both constructions is that these extensions will not inherit all the nice properties of the before mentioned system, e.g., the system introduced by Pap M. and Schipp F. is not analytic in the quaternionic setting. The system introduced by Qian T., Sprossig W., Wang J., is monogenic but can not be written in closed form.

Problem: Quaternionic analytic functions

- **R. Fueter-quaternionic analysis-1936**–extension based on Cauchy-Rieman equations
- **G. Gentili, D. C. Struppa–2006–2007** A new theory of regular functions of a quaternionic variable–Extension based on power series expansions

Slice regular functions

Set $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ to be the 2-sphere of purely imaginary units in $\mathbb{H}$, and for $\mathbf{I} \in \mathbb{S}$ let $L_{\mathbf{I}}$ be the complex plane $\mathbb{R} + \mathbb{R}\mathbf{I}$, then we have

$$\mathbb{H} = \cup_{\mathbf{I} \in \mathbb{S}} L_{\mathbf{I}}.$$

Definition Let A function $f : \mathbb{D}_{\mathbb{H}} \to \mathbb{H}$ is said to be (slice) regular if, for all $\mathbf{I} \in \mathbb{S}$, its restriction $f_{\mathbf{I}}$ to $\mathbb{D}_{\mathbb{H}_{\mathbf{I}}}$ is holomorphic, i.e., it has continuous partial derivatives and satisfies

$$\overline{\partial}_I f(x + y\mathbf{I}) := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{I}\frac{\partial}{\partial y}\right) f_{\mathbf{I}}(x + y\mathbf{I}) = 0. \tag{1}$$

Splitting Lemma. If f is a regular function on $\mathbb{D}_{\mathbb{H}}$, then for every $\mathbf{I} \in \mathbb{S}$ and for every $\mathbf{J} \in \mathbb{S}$, $\mathbf{J}$ orthogonal to $\mathbf{I}$, there exist two holomorphic functions $F, G : \mathbb{D}_{\mathbb{H}_{\mathbf{I}}} \to L_{\mathbf{I}}$, such that for every $z = x + y\mathbf{I} \in \mathbf{I}$, we have

$$f_{\mathbf{I}}(z) = F(z) + G(z)\mathbf{J}. \tag{2}$$

Slice regular functions

This recent theory has been growing very fast and was developed in a series of papers. The detailed up-to-date theory appears in the monograph **G. Gentili, C. Stoppato, D. C. Struppa**, Regular functions of a quaternionic variable, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.

On the open unit ball $\mathbb{D}_{\mathbb{H}}$, the class of regular functions coincides with the class of convergent power series of type $\sum_{n\geq 0} q^n a_n$, with all $a_n \in \mathbb{H}$. The direct extension of the Blaschke function, presented before, is not slice regular. In general the product and composition of two slice regular functions is not slice regular.

Slice regular product and composition

Definition Let $f, g : \mathbb{D}_{\mathbb{H}} \to \mathbb{H}$ be regular functions and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$; $g(q) = \sum_{n\in\mathbb{N}} q^n b_n$ be their power series expansions. The regular product of f and g ($*$-product) is the regular function defined by

$$f * g(q) = \sum_{n\in\mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k}$$

on the same ball $\mathbb{D}_{\mathbb{H}}$. The induced n-th power by $*$ product will be denoted by f^{*n}. The $*$-composition of f and g is given by

$$f(*g)(q) = \sum_{n\in\mathbb{N}} g(q)^{*n} a_n.$$

From the definition of the $*$-product and $*$-composition follows that the $*$-product and $*$-composition of two slice regular functions will be also slice regular. We can define two additional operations on regular functions.

Slice regular inverse

Definition Let $f : \mathbb{D}_{\mathbb{H}} \to \mathbb{H}$ be a regular function and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$ be its power series expansion. The regular conjugate of f is the regular function defined by $f^c(q) = \sum_{n\in\mathbb{N}} q^n \overline{a_n}$ on the same ball $\mathbb{B}$. The symmetrization of f is the function $f^s = f * f^c = f^c * f$.

Definition Let f be a regular function on a symmetric slice domain Ω. If $f \neq 0$ on Ω, the regular reciprocal of f is the function $f^{-*} = (f^s)^{-1} f^c$.

Slice regular Blaschke function

Definition The regular Blaschke function by definition is:

$$\mathcal{B}_a(q) = (q - a) * (1 - q\bar{a})^{-*}. \tag{3}$$

This function inherits all the nice properties of the complex Blaschke functions, i.e., is a regular fractional transformations that maps the open quaternionic unit ball $\mathbb{D}_{\mathbb{H}}$ onto itself and the boundary of unit ball $\mathbb{T}_{\mathbb{H}}$ onto itself bijectively.

C. Stoppato, Regular Moebius transformations of the space of quaternions, Ann. Global Anal. Geom., 39 (2010), 387-401.

C. Bisi an d C. Stoppato, Regular vs. Classical Möbius Transformations of the Quaternionic Unit Ball, Chapter Advances in Hypercomplex Analysis Volume 1 of the series Springer INdAM Series (2013), 1-13.

Properties of slice regular Blaschke functions

The classical and regular Blaschke functions are related in the following way:

$$\mathcal{B}_a(q) = (1 - q\bar{a})^{-*} * (q - a) = B_a(T_a(q)),$$

where $T_a(q) = (1 - qa)^{-1} q (1 - qa)$ is a diffeomorphism of $\mathbb{D}_{\mathbb{H}}$.
It can also be proved that the factors in the definition of the regular Blaschke product commute

$$\mathcal{B}_a(q) = (1 - q\bar{a})^{-*} * (q - a) = (q - a) * (1 - q\bar{a})^{-*}.$$

When $q, a \in L_{\mathbf{I}}$, then $\mathcal{B}_a(q) = B_a(q)$ and the slice regular composition of these functions on $L_{\mathbf{I}}$ is equal to the ordinary function composition.

Slice regular Hardy spaces

In analogy with the complex case, the slice regular Hardy space of quaternionic unit ball $H^2(\mathbb{D}_{\mathbb{H}})$ is the set of all functions $f(q) = \sum_{n\geq 0} q^n a_n$ for which

$$\|f\|^2 = \sum_{n\geq 0} |a_n|^2 < \infty. \tag{4}$$

The inner product on the space $H^2(\mathbb{D}_{\mathbb{H}})$ can be computed in two ways: if $f, g \in H^2(\mathbb{D}_{\mathbb{H}})$, let $f(q) = \sum_{n\geq 0} q^n a_n$, $g(q) = \sum_{n\geq 0} q^n b_n$ be their power series expansions, then their inner product is

$$\langle f, g\rangle = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \overline{g(re^{\mathbf{I}\theta})} f(re^{\mathbf{I}\theta}) d\theta = \sum_{n\geq 0} \overline{b_n} a_n, \tag{5}$$

for any $\mathbf{I} \in \mathbb{S}$.

D. Alpay, F. Colombo, I. Sabadini (2012), **D. Alpay, F. Colombo, I. Sabadini**, Schur analysis in the hyperholomorphic setting, Springer 2016.

Slice regular Malmquist-Takenaka system

Let us consider a sequence $a = (a_1, a_2, \dots)$ of quaternions in the unit ball, i.e., $|a_n| < 1$, $(n \in \mathbb{N}^*)$. The **slice regular analogue of the Malmquist-Takenaka system** can be expressed by the slice regular quaternionic Blaschke-functions:

$$\mathbf{\Phi}_1(z) = \sqrt{1 - |a_1|^2}(1 - z\overline{a_1})^{-*},$$

$$\mathbf{\Phi}_n(z) = \sqrt{1 - |a_n|^2}\left(* \prod_{k=1}^{n-1} \mathcal{B}_{a_k}(z) \right) * (1 - z\overline{a_n})^{-*} \ \ (z \in \mathbb{B},\ n = 2, 3, \dots). \tag{6}$$

Theorem 4. (PM 2017 arxiv.1611.06037) If all the parameters of the slice regular Malmquist -Takenaka system are on the same slice, i.e., there exists $I \in \mathbb{S}$ such that $a_n = r_n e^{\theta_n I} = r_n(\cos\theta_n + I \sin\theta_n)$, then the system $(\mathbf{\Phi}_n, n = 1, 2, \cdot)$ is a slice regular **complete orthonormal** system in $H^2(\mathbb{D}_{\mathbb{H}})$.

The properties of the projection operator

Let us consider the orthogonal projection operator of an arbitrary function $f \in H^2(\mathbb{B})$ on the subspace V_n spanned by the functions $\{\boldsymbol{\Phi}_k,\ k = 1, \cdots, n\}$

$$P_n f(z) = \sum_{k=1}^{n} \boldsymbol{\Phi}_k(z)\langle f, \boldsymbol{\Phi}_k\rangle, \tag{7}$$

where the value of the scalar product $\langle f, \boldsymbol{\Phi}_k\rangle$ is

$$\langle f, \boldsymbol{\Phi}_k\rangle = \lim_{r\to 1_-} \frac{1}{2\pi}\int_0^{2\pi} \overline{\boldsymbol{\Phi}_k(re^{I\theta})} f(re^{I\theta})d\theta =$$

$$\lim_{r\to 1_-} \frac{1}{2\pi}\int_0^{2\pi} \overline{\boldsymbol{\Phi}_k(re^{I\theta})} F(re^{I\theta})d\theta + \lim_{r\to 1_-} \frac{1}{2\pi}\int_0^{2\pi} \overline{\boldsymbol{\Phi}_k(re^{I\theta})} G(re^{I\theta})d\theta J.$$

If $\sum_{n\geq 0}(1 - |a_n|) = +\infty$, then the system $\boldsymbol{\Phi}_n, (n \in \mathbb{N}^*)$ is complete in $H^2(\mathbb{B})$, this implies that for every $f \in H^2(\mathbb{B})$ the projection of f on V_n converges in norm to f.

The properties of the projection operator

Theorem 5. (PM 2017 arxiv.1611.06037) If the parameters of the slice regular Malmquist -Takenaka system are on the same slice, i.e., there exists $I \in \mathbb{S}$ such that $a_n = r_n e^{\theta_n I}$ ($r_n < 1$, $n \in \mathbb{N}^*$), then for all $f \in H^2(\mathbb{B})$ the restriction of the projection operator $P_n f$ to the slice $\mathbb{B}_I$ of the unit ball is an interpolation operator in the points $a_\ell = r_\ell e^{\theta_\ell I}$ ($\ell \in \{1, \cdots, n\}$).

References

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THANK YOU FOR YOUR ATTENTION