Images of some subspaces of $L^2(\mathbb{R}^m)$ under Grushin and Hermite semigroup

Partha Sarathi Patra

(This is a joint work with Dr. D Venku Naidu)

Department of Mathematics
Indian Institute of Technology Hyderabad, India

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

6^{th} Fourier Analysis Workshop in Fourier Analysis and Related Area, August 24-31, 2017

Outline

Definition

Bargmann space or Fock space $\mathcal{F}$ is the Hilbert space of entire function on $\mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 d\lambda(z) < \infty$$

where $d\lambda(z) = \frac{1}{\pi} e^{-|z|^2} dxdy$, and the inner-product is defined by $\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda(z)$.

Definition

Bargmann space or Fock space $\mathcal{F}$ is the Hilbert space of entire function on $\mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 d\lambda(z) < \infty$$

where $d\lambda(z) = \frac{1}{\pi} e^{-|z|^2} dxdy$, and the inner-product is defined by $\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda(z)$.

Theorem

[1] The Bargmann transform

$$\mathcal{B}f(z) = \int_{\mathbb{R}} f(x) e^{2xz - x^2 - \frac{1}{2}z^2} dx$$

is an isometry from $L^2(\mathbb{R})$ onto $\mathcal{F}$.

Outline

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $k \in \mathbb{N}_0$, then the normalized Hermite function, related to k^{th} degree Hermite polynomial in $\mathbb{R}$ is,

$$h_k(x) = (2^k k! \pi^{\frac{1}{2}})^{\frac{-1}{2}} (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{\frac{x^2}{2}},$$

and n-dimensional normalized Hermite function ϕ_α is given by

$$\phi_\alpha(x) = \prod_{i=1}^{n} h_{\alpha_i}(x), \text{ where } \alpha = (\alpha_i)_{i=1}^{n} \in \mathbb{N}_0^n.$$

- Hermite Operator: $H = -\Delta_x + x^2$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $k \in \mathbb{N}_0$, then the normalized Hermite function, related to k^{th} degree Hermite polynomial in $\mathbb{R}$ is,

$$h_k(x) = (2^k k! \pi^{\frac{1}{2}})^{\frac{-1}{2}} (-1)^k \frac{d^k}{dx^k}(e^{-x^2}) e^{\frac{x^2}{2}},$$

and n-dimensional normalized Hermite function ϕ_α is given by

$$\phi_\alpha(x) = \prod_{i=1}^{n} h_{\alpha_i}(x), \text{ where } \alpha = (\alpha_i)_{i=1}^{n} \in \mathbb{N}_0^n.$$

- Hermite Operator: $H = -\Delta_x + x^2$
- ϕ_α's are eigen vector of H corresponding to the eigen value $(2|\alpha| + n)$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $k \in \mathbb{N}_0$, then the normalized Hermite function, related to k^{th} degree Hermite polynomial in $\mathbb{R}$ is,

$$h_k(x) = (2^k k! \pi^{\frac{1}{2}})^{\frac{-1}{2}} (-1)^k \frac{d^k}{dx^k}(e^{-x^2}) e^{\frac{x^2}{2}},$$

and n-dimensional normalized Hermite function ϕ_α is given by

$$\phi_\alpha(x) = \prod_{i=1}^{n} h_{\alpha_i}(x), \text{ where } \alpha = (\alpha_i)_{i=1}^n \in \mathbb{N}_0^n.$$

- Hermite Operator: $H = -\Delta_x + x^2$
- ϕ_α's are eigen vector of H corresponding to the eigen value $(2|\alpha| + n)$
- Fourier transform $\mathbb{R}^n$: $\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$,

Outline

Grushin Semigroup

- Grushin operator on $\mathbb{R}^{n+1}$:$G = -(\Delta_x + |x|^2 \frac{\partial^2}{\partial t^2})$,
- Heat Equation Corresponding to this Grushin operator:

$$\frac{\partial}{\partial s} u(x, t; s) = -Gu(x, t; s), \tag{2.1}$$

  with the initial condition $u(x, t, 0) = f(x, t)$ where f is a function in $L^2(\mathbb{R}^{n+1})$
- The solution: $u(x, t; s) = e^{-sG} f(x, t)$.

Grushin Semigroup

- Grushin operator on $\mathbb{R}^{n+1}$:$G = -(\Delta_x + |x|^2 \frac{\partial^2}{\partial t^2})$,
- Heat Equation Corresponding to this Grushin operator:

$$\frac{\partial}{\partial s} u(x,t;s) = -Gu(x,t;s), \tag{2.1}$$

  with the initial condition $u(x,t,0) = f(x,t)$ where f is a function in $L^2(\mathbb{R}^{n+1})$
- The solution: $u(x,t;s) = e^{-sG} f(x,t)$.

Grushin Semigroup

- Grushin operator on $\mathbb{R}^{n+1}$:$G = -(\Delta_x + |x|^2 \frac{\partial^2}{\partial t^2})$,
- Heat Equation Corresponding to this Grushin operator:

$$\frac{\partial}{\partial s} u(x, t; s) = -Gu(x, t; s), \tag{2.1}$$

  with the initial condition $u(x, t, 0) = f(x, t)$ where f is a function in $L^2(\mathbb{R}^{n+1})$
- The solution: $u(x, t; s) = e^{-sG} f(x, t)$.

Grushin Semigroup

- Grushin operator on $\mathbb{R}^{n+1}$: $G = -(\Delta_x + |x|^2 \frac{\partial^2}{\partial t^2})$,
- Heat Equation Corresponding to this Grushin operator:

$$\frac{\partial}{\partial s} u(x, t; s) = -Gu(x, t; s), \tag{2.1}$$

  with the initial condition $u(x, t, 0) = f(x, t)$ where f is a function in $L^2(\mathbb{R}^{n+1})$
- The solution: $u(x, t; s) = e^{-sG} f(x, t)$.
- Fourier transform with respect to last variable Notation: $f^\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x, t) e^{-i\lambda t} dt$

Grushin Semigroup

- Grushin operator on $\mathbb{R}^{n+1}$:$G = -(\Delta_x + |x|^2 \frac{\partial^2}{\partial t^2})$,
- Heat Equation Corresponding to this Grushin operator:

$$\frac{\partial}{\partial s} u(x,t;s) = -Gu(x,t;s), \tag{2.1}$$

with the initial condition $u(x,t,0) = f(x,t)$ where f is a function in $L^2(\mathbb{R}^{n+1})$

- The solution: $u(x,t;s) = e^{-sG} f(x,t)$.
- Fourier transform with respect to last variable Notation: $f^\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x,t) e^{-i\lambda t} dt$
- Parametrized hermite function: $\phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}} \phi_\alpha(\sqrt{|\lambda|}x)$,

- $\lambda \neq 0$ then $\{\phi_\alpha^\lambda\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$ and

$$H_\lambda \phi_\alpha^\lambda(x) = (2|\alpha| + n)|\lambda|\phi_\alpha^\lambda(x).$$

- Parametraized Hermite operator: $H_\lambda = -\Delta_x + |x|^2\lambda^2$, $\lambda \neq 0$
- Fourier transform on the last variable reduces the Grushin heat equation to

$$\frac{\partial}{\partial s} u^\lambda = -H_\lambda u^\lambda$$

with initial condition $u^\lambda(x;0) = f^\lambda(x)$.

- With the help of spectral resolution of H_λ we have

$$e^{-sH_\lambda}f(x) = \sum_\alpha e^{-(2|\alpha|+n)|\lambda|s}\langle f,\phi_\alpha^\lambda\rangle_{L^2(\mathbb{R}^n)}\phi_\alpha^\lambda(x).$$
$$= \int_{\mathbb{R}^n} K_s^\lambda(x,y)f(y)dy \text{ (using Mehlar's formula).}$$

where

$$K_s^\lambda(x,y)$$
$$=(2\pi)^{\frac{-n}{2}}\left(\frac{|\lambda|}{\sinh(2s|\lambda|)}\right)^{\frac{n}{2}} e^{\frac{-|\lambda|}{2}(x^2+y^2)\coth(2s|\lambda|)} e^{\frac{|\lambda|xy}{\sinh(2|\lambda|s)}}.$$

- $$\begin{aligned} e^{-sG}f(x,t) &= u(x,t;s) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u^{\lambda}(x;s)e^{i\lambda t}d\lambda. \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\lambda t}\int_{\mathbb{R}^n} K_s^{\lambda}(x,y)f^{\lambda}(y)dyd\lambda \end{aligned}$$
- $e^{-sG}f(x,t)$ can be extended to a holomorphic function in both the variables.
- $e^{-sG} : L^2(\mathbb{R}^{n+1}) \longrightarrow \mathcal{O}(\mathbb{C}^{n+1})$, where $\mathcal{O}(\mathbb{C}^{n+1})$ is the vector space of holomorphic functions on $\mathbb{C}^{n+1}$.

Let us consider the Hilbert space

$$L^2\widetilde{(\mathbb{R}^{n+1})} = \left\{ f \in L^2(\mathbb{R}^{n+1}) : \int_{\mathbb{R}^{n+1}} |f^\lambda(x)|^2 e^{\lambda^2} dx d\lambda < \infty \right\},$$

where the inner product is $\langle f, g \rangle = \int_{\mathbb{R}^{n+1}} f^\lambda(x) \overline{g^\lambda(x)} e^{\lambda^2} dx d\lambda$, We wish to find a positive weight function $W_s(z, w)$ where $z \in \mathbb{C}^n$ and $w \in \mathbb{C}$ such that,

$$\int_{\mathbb{C}^{n+1}} |e^{-sG} f(z, w)|^2 W_s(z, w) dz dw = \|f\|^2_{L^2\widetilde{(\mathbb{R}^{n+1})}}. \qquad (2.2)$$

- we note that if $u(x,t;s)$ is solution of (**??**) with initial condition from $\widetilde{L^2(\mathbb{R}^{n+1})}$, then for each $x \in \mathbb{R}^n$, $u(x,\cdot\,;\,\cdot)$ is solution to the 1-dimensional heat equation with initial condition from the following space,

$$\widetilde{L^2(\mathbb{R})} = \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2 e^{\lambda^2} d\lambda \right\}, \tag{2.3}$$

- i.e, for each x, $u(x,t;s) = h_s(f)(t)$ with $f \in \widetilde{L^2(\mathbb{R})}$, where h_s is the heat kernel transform

So we will find first the image of h_s in the following sub-section,

Outline

Let

$$q_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

For $f \in \widetilde{L^2(\mathbb{R})}$, $f * q_t(x)$ has holomorphic extension on $\mathbb{C}$, where $f * q_t(x) = \int_{\mathbb{R}} f(y) q_t(y-x) dy$. The heat kernel tranform $h_t : \widetilde{L^2(\mathbb{R})} \longrightarrow \mathcal{O}(\mathbb{C})$ such that $f \longmapsto f * q_t$, is one to one.

Lemma

$h_t(\widetilde{L^2(\mathbb{R})})$ is reproducing kernel Hilbert space with kernel

$$\mathcal{K}_z(w) = C_t e^{-\frac{\bar{z}^2 + w^2}{4(2t+1)} + \frac{\bar{z}w}{2(2t+1)}}$$

where $C_t = \frac{1}{\sqrt{2(2t+1)}}$ is a constant depending on t.

Proposition

For $F \in h_t(\widetilde{L^2(\mathbb{R})})$, $|F(x+iy)| \leq \sqrt{C_t}\|F\|_{h_t(\widetilde{L^2(\mathbb{R})})} e^{\frac{y^2}{2(2t+1)}}$.

Proposition

For $F \in h_t(\widetilde{L^2(\mathbb{R})})$, $|F(x+iy)| \leq \sqrt{C_t}\|F\|_{h_t(\widetilde{L^2(\mathbb{R})})} e^{\frac{y^2}{2(2t+1)}}$.

- This gives a growth condition for the elements of $h_t(\widetilde{L^2(\mathbb{R})})$

Proposition

For $F \in h_t(\widetilde{L^2(\mathbb{R})})$, $|F(x+iy)| \leq \sqrt{C_t}\|F\|_{h_t(\widetilde{L^2(\mathbb{R})})} e^{\frac{y^2}{2(2t+1)}}$.

- This gives a growth condition for the elements of $h_t(\widetilde{L^2(\mathbb{R})})$
- Let us consider the following Hilbert Space of holomorphic functions

$$\mathcal{B}_{2t+1}(\mathbb{C})$$

$$=\left\{ f : \mathbb{C} \overset{holomorphic}{\longrightarrow} \mathbb{C} : \|f\|^2 = \sqrt{\frac{4\pi}{2t+1}} \int_{\mathbb{C}} |f(z)|^2 e^{\frac{-y^2}{2t+1}} dxdy < \infty \right\}.$$

Proposition

Let $e_m(z) = z^m e^{\frac{-z^2}{4(2t+1)}}$*, then the set* $A = \{e_m : m = 0, 1, 2, \cdots\}$ *is a complete orthogonal set in* $\mathcal{B}_{2t+1}(\mathbb{C})$*. Moreover* $A \subset \mathcal{B}_{2t+1}(\mathbb{C}) \cap h_t(\widetilde{L^2(\mathbb{R})})$.

Theorem

The image of $\widetilde{L^2(\mathbb{R})}$ under heat kernel transform is the space of holomorphic functions $\mathcal{B}_{2t+1}(\mathbb{C})$.

Let's define $\widetilde{L^2_+(\mathbb{R})} = \{f \in \widetilde{L^2(\mathbb{R})} : \mathcal{F}f(\lambda) = 0 \text{ when } \lambda \leq 0\}$ and $\widetilde{L^2_-(\mathbb{R})} = \{f \in \widetilde{L^2(\mathbb{R})} : \mathcal{F}f(\lambda) = 0 \text{ when } \lambda \geq 0\}$, then since $\widetilde{L^2(\mathbb{R})} = \widetilde{L^2_+(\mathbb{R})} \oplus \widetilde{L^2_-(\mathbb{R})}$ we can observe that,

$$\mathcal{B}_{2t+1}(\mathbb{C}) := h_t(\widetilde{L^2_+(\mathbb{R})}) \oplus h_t(\widetilde{L^2_-(\mathbb{R})}) = \mathcal{B}^+_{2t+1}(\mathbb{C}) \oplus \mathcal{B}^-_{2t+1}(\mathbb{C}).$$

Theorem

[11] Let $s > 0$, $\lambda \neq 0$ and

$$U_s^\lambda(z) = 4^n \left(\frac{|\lambda|}{\sinh 4s|\lambda|} \right)^{\frac{n}{2}} e^{|\lambda|(\tanh(2s|\lambda|)x^2 - \coth(2|\lambda|s)y^2)}.$$

Then the semigroup e^{-sH_λ} is isometric isomorphism from $L^2(\mathbb{R}^n)$ to $H_s^\lambda(\mathbb{C}^n)$ where, $H_s^\lambda(\mathbb{C}^n)$ is the space of holomorphic functions on $\mathbb{C}^n$ such that, $\|F\|^2 = \int_{\mathbb{C}^n} |F(z)|^2 U_s^\lambda(z) dz < \infty$.

Lemma

$W_s(z,w)$ satisfying the equation

$$\int_{\mathbb{C}^{n+1}} |e^{-sG} f(z,w)|^2 W_s(z,w) dz dw = \|f\|^2_{L^2(\widetilde{\mathbb{R}^{n+1}})}. \tag{3.1}$$

with $w = \xi + i\eta$ is independent of ξ.

Lemma

For each $\lambda_0 \neq 0$, the non-negative weight function in (3.1) satisfies the following,

$$U_s^{\lambda_0}(z) = \int_{\mathbb{R}} e^{-2\eta\lambda_0} W_s(z, i\eta) e^{-\lambda_0^2} d\eta \tag{3.2}$$

where $U_s^{\lambda_0}(z)$ is the weight of λ_0-parametrized Hermite Bergman space.

Theorem

There is no non-negative weight function $W_s(z,w)$ satisfying (3.1).

- So it's not possible to characterize the image of Grushin semigroup as non-negative weighted Bergman space with isometry.
- Now we take $\lambda > 0$ and consider the following formal expression,

$$W_s^+(z,w) = \int_{\mathbb{R}} e^{(\lambda+\frac{is}{2})^2} U_s^{\lambda+\frac{is}{2}}(z) e^{2\eta(\lambda+\frac{is}{2})} ds. \tag{3.3}$$

Proposition

The function $W_s^+(z, \xi + i\eta)$ *defined in the above equation* (3.3) *is well defined everywhere, independent of choice of* ξ *and having following properties,*

1. W_s^+ *is independent of the choice of* λ*, in fact*

$$W_s^+(z, \xi + i\eta) = \lim_{\lambda \to 0^+} \int_{\mathbb{R}} e^{(\lambda + \frac{is}{2})^2} U_s^{\lambda + \frac{is}{2}}(z) e^{2\eta(\lambda + \frac{is}{2})} ds \quad (3.4)$$

2. *Let* $a > 0$ *and* $Q \subseteq \mathbb{C}^n$ *be compact set. Then there exist a constant* $C = C(Q, a) > 0$ *such that for all* $\epsilon \in [a^{-1}, a]$ *and* $\xi \in \mathbb{R}$ *we have,*

$$\sup_{z \in Q} \int_{\mathbb{R}} |e^{2\epsilon\eta} W_s^+(z, \xi + i\eta)| d\eta < C. \quad (3.5)$$

3. W_s^+ *satisfies* (3.2) *with* $\lambda > 0$*, i.e.*

$$U_s^{\lambda}(z) = \int_{\mathbb{R}} e^{-2\eta\lambda} W_s^+(z, i\eta) e^{-\lambda^2} d\eta. \quad (3.6)$$

Let's introduce the following subspaces of $L^2(\widetilde{\mathbb{R}^{n+1}})$,

$$L^2_+(\widetilde{\mathbb{R}^{n+1}}) = \{f \in L^2(\widetilde{\mathbb{R}^{n+1}}) : f^\lambda = 0, \text{ when } \lambda \leq 0\} \text{ and}$$

$$L^2_-(\widetilde{\mathbb{R}^{n+1}}) = \{f \in L^2(\widetilde{\mathbb{R}^{n+1}}) : f^\lambda = 0, \text{ when } \lambda \geq 0\}.$$

Then we have the decomposition,
$L^2(\widetilde{\mathbb{R}^{n+1}}) = L^2_+(\widetilde{\mathbb{R}^{n+1}}) \oplus L^2_-(\widetilde{\mathbb{R}^{n+1}})$.
Let $R > 0$, B_R denotes the ball in $\mathbb{C}^n$ of radius R, centered at 0, and define $K_R = B_R \times \mathbb{C}$, then $\cup_{R>0} K_R = \mathbb{C}^{n+1}$.
Now we define the vector space $\mathcal{V}_s^+(\mathbb{C}^{n+1})$ containing all holomorphic functions F on $\mathbb{C}^{n+1}$ such that the followings hold,

1. $F|_{K_R} \in L^2(K_R, |W_s^+(z,w)|dzdw)$,
2. $\lim_{R\to\infty} \int_{K_R} |F(z,w)|^2 W_s^+(z,w)dzdw < \infty$, and
3. $F(z,\cdot) \in \mathcal{B}^+_{2s+1}(\mathbb{C})$.

And define the sesquilinear form on $\mathcal{V}_s^+(\mathbb{C}^{n+1})$ by,

$$\langle F, G\rangle_+ = \lim_{R\to\infty} \int_{K_R} F(z,w)\overline{G(z,w)}W_s^+(z,w)dzdw, \qquad (3.7)$$

where $F, G \in \mathcal{V}_s^+(\mathbb{C}^{n+1})$.

Lemma

The sesquilinear form $\langle \cdot, \cdot \rangle_+$ *induces an innerproduct on* $\mathcal{V}_s^+(\mathbb{C}^{n+1})$.

Theorem

Let $s > 0$, $\mathcal{B}_s^+(\mathbb{C}^{n+1}) := e^{-sG}(\widetilde{L_+^2(\mathbb{R}^{n+1})})$ *is the Hilbert completion of* $\mathcal{V}_s^+(\mathbb{C}^{n+1})$ *with respect to the inner product* $\langle , \rangle_+$ *defined in* (3.7). *In fact,* $\mathcal{B}_s^+(\mathbb{C}^{n+1})$ *is isometrically isomorphic to the completion of* $\mathcal{V}_s^+(\mathbb{C}^{n+1})$.

On the similar line of (3.3) we can define $W_s^-(z,w)$ when $\lambda < 0$, then a proposition like (3.4) is valid, and as well $\mathcal{V}_s^-(\mathbb{C}^{n+1})$ can be constructed as above, consequently $\mathcal{B}_s^-(\mathbb{C}^{n+1}) = e^{-sG}(\widetilde{L^2_-(\mathbb{R}^{n+1})})$ is Hilbert completion of $\mathcal{V}_s^-(\mathbb{C}^{n+1})$ will be followed. And hence we have the direct sum decomposition

$$e^{-sG}(\widetilde{L^2(\mathbb{R}^{n+1})}) = \mathcal{B}_s^+(\mathbb{C}^{n+1}) \oplus \mathcal{B}_s^-(\mathbb{C}^{n+1}). \tag{3.8}$$

Hermite-Sobolev space of positive order

Definition

For $\mu > 0$ the Hermite Soblolev space $W_H^{\mu,2}(\mathbb{R}^n)$ is defined as the image of $L^2(\mathbb{R}^n)$ under $H^{-\mu}$. That is $f \in W_H^{\mu,2}(\mathbb{R}^n)$ if and only if

$$\|f\|_{W_H^{\mu,2}} := \sum_{\alpha} (2|\alpha| + n)^{2\mu} |\langle f, \phi_\alpha \rangle|^2 < \infty.$$

- $W_H^{\mu,2}(\mathbb{R}^n)$ forms a Hilbert space under the inner product

$$\langle f, g \rangle := \sum_{\alpha} (2|\alpha| + n)^{2\mu} \langle f, \phi_\alpha \rangle \overline{\langle g, \phi_\alpha \rangle}.$$

- Define holomorphic Sobolev space
  $W_t^{\mu,2}(\mathbb{C}^n) := e^{-tH}(W_H^{\mu,2}(\mathbb{R}^n))$, $t > 0$.
  this is a Hilbert space with the inner product,
  $\langle F, G \rangle_{W_t^{\mu,2}} := \langle f, g \rangle_{W_H^{\mu,2}}$, where $e^{-tH} f = F$ and $e^{-tH} g = G$.

- Clearly e^{-tH} is an isometric isomorphism from $W_H^{\mu,2}(\mathbb{R}^n)$ to the holomorphic space $W_t^{\mu,2}(\mathbb{C}^n)$.
- Let us define $\mathcal{F}_t^{\mu}$, vector space of holomorphic functions F on $\mathbb{C}^n$ such that,

$$\int_{\mathbb{C}^n} |F(z)|^2 \int_0^t (t-s)^{m-2\mu-1} \left| \frac{d^m}{ds^m}(U_s(z)) \right| dsdz < \infty.$$

- We use following Caputo fractional derivative

$$C_* D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(x)}{(t-x)^{\alpha+1-m}} dx,$$

where m is the positive integer with $m-1 < \alpha < m$ and $f^{(m)}$ is the ordinary derivative of order m.

Image of Hermite Sobolev space under e^{-tH}

Lemma

$\mathcal{F}_t^\mu$ is a pre-Hilbert space with respect to the sesquilinear form, $\langle F, G\rangle = \int_{\mathbb{C}^n} F(z)\overline{G(z)} C_* D_t^{2\mu}(U_t(z))dz$. Moreover there exist constants $C_1,\ C_2 > 0$ such that,

$$C_1 \left\| F \right\|_{W_t^{\mu,2}} \leq \left\| F \right\|_{\mathcal{F}_t^\mu} \leq C_2 \left\| F \right\|_{W_t^{\mu,2}}, \tag{4.1}$$

when $F = e^{-tH} f$ with f in Schwartz space.

Theorem

The holomorphic Sobolev space $W_t^{\mu,2}(\mathbb{C}^n)$ can be identified as the completion of $\mathcal{F}_t^\mu(\mathbb{C}^n)$.

Grushin Sobolev space Of Positive Order

Definition

For $s > 0$, we define Grushin Sobolev space $\widetilde{W_G^{\mu,2}}(\mathbb{R}^{n+1})$ as $\widetilde{\text{sub-space of } L^2(\mathbb{R}^{n+1})}$ such that an element f satisfies the following condition,

$$\int_{-\infty}^{\infty} \sum_{\alpha} (2|\alpha| + n)^{2\mu} |\lambda|^{2\mu} |\langle f^{\lambda}, \phi_{\alpha}^{\lambda} \rangle|^2 e^{\lambda^2} d\lambda < \infty. \tag{5.1}$$

- We denote $\widetilde{W_{G+}^{\mu,2}}(\mathbb{R}^{n+1})$ as the Hilbert space of all those $f \in \widetilde{W_G^{\mu,2}}(\mathbb{R}^{n+1})$ such that, $\int_0^{\infty} \sum_{\alpha} (2|\alpha| + n)^{2\mu} |\lambda|^{2\mu} |\langle f^{\lambda}, \phi_{\alpha}^{\lambda} \rangle|^2 e^{\lambda^2} d\lambda < \infty$, The space $\widetilde{W}^{\mu,2}_{s,+}(\mathbb{C}^{n+1}) := e^{-sG}(\widetilde{W}^{\mu,2}_{G,+}(\mathbb{R}^{n+1}))$ is made into a Hilbert space by defining the innerproduct in following way,

$$\begin{aligned} \langle F, G\rangle_{\widetilde{W}^{\mu,2}_{s,+}(\mathbb{C}^{n+1})} &= \langle f, g\rangle_{\widetilde{W}^{\mu,2}_{G,+}(\mathbb{R}^{n+1})} \\ &= \int_0^\infty \sum_\alpha (2|\alpha| + n)^{2\mu} |\lambda|^{2\mu} \langle f^\lambda, \phi^\lambda_\alpha\rangle \overline{\langle \phi^\lambda_\alpha, g^\lambda\rangle} e^{\lambda^2} d\lambda, \end{aligned}$$

where $F = e^{-sG} f$ and $G = e^{-sG} g$. For $\lambda > 0$, let us consider the weight function

$$W_s^{+,\mu}(z,w) = \int_{\mathbb{R}} e^{\left(\lambda + \frac{is}{2}\right)^2} C_* D_s^{2\mu}\left(U_s^{\lambda+\frac{is}{2}}(z)\right) e^{2\eta\left(\lambda+\frac{is}{2}\right)} ds. \quad (5.2)$$

Then this weight obeys all properties similar to the proposition (3.4). Define the vector space $\mathcal{V}_s^{+,\mu}(\mathbb{C}^{n+1})$ of holomorphic functions F on $\mathbb{C}^{n+1}$ such that

1. For all $R > 0$,

$$\int_{K_R}\int_{\mathbb{R}}\int_0^s |F(z,w)|^2 |e^{(\lambda+\frac{is}{2})^2}\frac{d^m}{d\theta^m}U_\theta^{(\lambda+\frac{is}{2})}(z)e^{2\eta(\lambda+\frac{is}{2})}|dsd\theta dzdw < \infty \tag{5.3}$$

2. $\lim_{R\to\infty}\int_{K_R}|F(z,w)|^2 W_s^{+,\mu}(z,w)dzdw < \infty$
3. $F(z,\cdot) \in \mathcal{B}_{2s+1}(\mathbb{C})$ for all $z \in \mathbb{C}^n$.

And the sesquilinear form in $\mathcal{V}_s^{+,\mu}(\mathbb{C}^{n+1})$,

$$\langle F, G\rangle_{+,\mu} = \lim_{R\to\infty}\int_{K_R} F(z,w)\overline{g(z,w)}W_s^{+,\mu}(z,w)dzdw.$$

Proposition

The above sesqulinear form is an innerproduct and there exist positive constants C_1 and C_2 such that,

$$C_1 \|F\|_{\widetilde{W}^{\mu,2}_{s,+}(\mathbb{C}^{n+1})} \leq \|F\|_{\mathcal{V}^{+,\mu}_s(\mathbb{C}^{n+1})} \leq C_2 \|F\|_{\widetilde{W}^{\mu,2}_{s,+}(\mathbb{C}^{n+1})}, \tag{5.4}$$

where $F = e^{-G} f$, f is in suitable dense class of $\widetilde{W}^{\mu,2}_{G,+}(\mathbb{R}^{n+1})$.

Proposition

$\widetilde{W}^{\mu,2}_{s,+}(\mathbb{C}^{n+1})$ is the Hilbert completion of $\mathcal{V}^{+,\mu}_s(\mathbb{C}^{n+1})$.

When $\lambda < 0$, considering the weight function $W^{-,\mu}_s(z, w)$, similar way as above, we can construct $\mathcal{V}^{-,\mu}_s(\mathbb{C}^{n+1})$ and conclude that $\widetilde{W}^{\mu,2}_{s,-}(\mathbb{C}^{n+1})$ is Hilbert completion of $\mathcal{V}^{-,\mu}_s(\mathbb{C}^{n+1})$.

Theorem

The image of Grushin Sobolev space under e^{-sG} is identified as direct-sum of two weighted Bergman space. That is,

$$\widetilde{W}^{\mu,2}_s(\mathbb{C}^{n+1}) = \widetilde{W}^{\mu,2}_{s,+}(\mathbb{C}^{n+1}) \oplus \widetilde{W}^{\mu,2}_{s,-}(\mathbb{C}^{n+1}).$$

Theorem

The image of Grushin Sobolev space under e^{-sG} is identified as direct-sum of two weighted Bergman space. That is,

$$\widetilde{W}_{s}^{\mu,2}(\mathbb{C}^{n+1}) = \widetilde{W}_{s,+}^{\mu,2}(\mathbb{C}^{n+1}) \oplus \widetilde{W}_{s,-}^{\mu,2}(\mathbb{C}^{n+1}).$$

References I

[1] V. Bargmann, On a Hilbert space of analytic functions and an associated integral trans- form, Part I, Comm. Pure Appl. Math. 14 (1961), 187-214.

[2] A. Dasgupta, S. Molahajloo and M. W. Wong, The Inverse, the Heat Semigroup, Liouville's Theorems and the Spectrum for the Grushin Opera- tor, Journal of Pseudo-Differential Operators and Applications 1 (4) (2010), 377–388.

[3] B.C. Hall, The Segal-Bargmann "coherent state" transform for compact Lie groups, J. Funct. Anal. 122 (1994), no. 1, 103-151.

[4] B.C. Hall, W. Lewkeeratiyutkul, Holomorphic Sobolev spaces and the generalized Segal–Bargmann transform, J. Funct. Anal., 217 (2004), pp. 192–220.

References II

[5] Krötz, Bernhard; Thangavelu, Sundaram; Xu, Yuan The heat kernel transform for the Heisenberg group, J. Funct. Anal. 225 (2005), no. 2, 301–336.

[6] M. Stenzel, The Segal–Bargmann transform on symmetric space of compact type, J. Funct. Anal., 165 (1) (1999), pp. 44–58.

[7] Radha, R.; Thangavelu, S. Holomorphic Sobolev spaces, Hermite and special Hermite semigroups and a Paley-Wiener theorem for the windowed Fourier transform, J. Math. Anal. Appl. 354 (2009), no. 2, 564–574.

[8] S. Molahajloo and M. W. Wong, Diagonalization of Weyl Transforms and Heat Equations for Time-Dependent Heat Equations, Complex Analysis and Operator Theory 5 (1) (2011), 283–298.

References III

[9] S. Thangavelu, Holomorphic Sobolev spaces associated to compact symmetric spaces, J. Funct. Anal., 251 (2) (2007), pp. 438–462.

[10] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Princeton University Press (1993).

[11] S. Thangavelu, Hermite and Laguerre semigroups: some recent developments. Orthogonal families and semigroups in analysis and probability, 251–284, Sémin. Congr., 25, Soc. Math. France, Paris, 2012.

[12] S.Thangavelu, An introduction to the uncertainty principle. Hardy's theorem on Lie groups, Progress in Mathematics, 217. Birkhäuser Boston, Inc., Boston, MA, 2004. xiv+174 pp. ISBN: 0-8176-4330-3.

Thank you for listening