Spectral Synthesis on Affine Groups

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The Spectral Synthesis Theorem

Laurent Schwartz, 1947

Spectral synthesis holds on the reals. In other words: given any continuous complex valued function f on the reals it is the uniform limit on compact sets of linear combinations of exponential monomials of the form $x \mapsto x^n e^{\lambda x}$ (n is a natural number, λ is a complex number) such that all these exponential monomials belong to the smallest linear space including all translates of f and being closed with respect to uniform convergence on compact sets.

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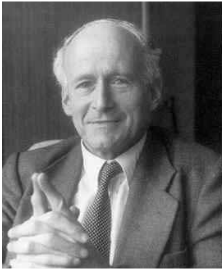

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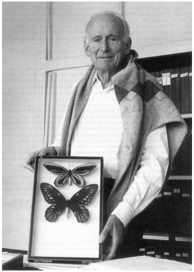

With his butterflies

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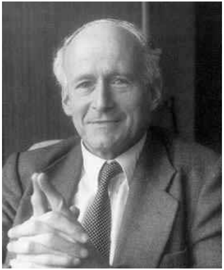

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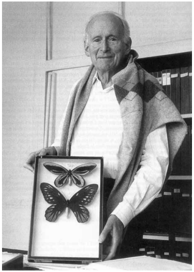

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Counterexamples

No direct extension of Schwartz's result to $\mathbb{R}^n$ is possible:

Spectral synthesis fails to hold in $\mathbb{R}^n$ for $n \geqslant 2$

(Dmitrii I. Gurevich, 1975) For each natural number $n \geqslant 2$ there exist compactly supported measures μ, ν such that the exponential monomial solutions of the system of functional equations

$$\mu * f = 0, \qquad \nu * f = 0$$

do not span a dense subspace in the solution space of this system.

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$$\Delta_{m;y} = \delta_{-y} - m(y)\delta_0,$$

the *modified difference* corresponding to m with increment y. Higher order differences:

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K: compact subgroup with normalized Haar measure ω

K-invariant functions in $\mathcal{C}(G)$: $f(kxl) = f(x)$ for x in G and k, l in K. These can be identified with the space $\mathcal{C}(G//K)$.

K-invariant measures in $\mathcal{M}_c(K)$: for each f in $\mathcal{C}(G)$

$$\int_G f(x)\, d\mu(x) = \int_G \int_K \int_K f(kxl)\, d\omega(k)\, d\omega(l)\, d\mu(x)$$

These can be identified with the functions in the space $\mathcal{M}_c(G//K)$, which can be identified with a closed subalgebra of $\mathcal{M}_c(K)$.

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K-invariant functions in $\mathcal{C}(G)$: $f(kxl) = f(x)$ for x in G and k, l in K. These can be identified with the space $\mathcal{C}(G//K)$.

K-invariant measures in $\mathcal{M}_c(K)$: for each f in $\mathcal{C}(G)$

$$\int_G f(x)\,d\mu(x) = \int_G \int_K \int_K f(kxl)\,d\omega(k)\,d\omega(l)\,d\mu(x)$$

These can be identified with the functions in the space $\mathcal{M}_c(G//K)$, which can be identified with a closed subalgebra of $\mathcal{M}_c(K)$.

The pair (G, K) is called a *Gelfand pair* if the algebra $\mathcal{M}_c(G//K)$ is commutative.

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Projection

The projection $f \mapsto f^{\#}$ on $\mathcal{C}(G)$ is defined as

$$f^{\#}(x) = \int_K \int_K f(kxl)\, d\omega(k)\, d\omega(l) \text{ for } x \in G.$$

The projection $\mu \mapsto \mu^{\#}$ on $\mathcal{M}_c(G)$ is defined as

$$\langle \mu^{\#}, f \rangle = \int_G f^{\#}(x)\, d\mu(x) \text{ for } f \in \mathcal{C}(G).$$

Then $f \mapsto f^{\#}$ is a continuous linear mapping from $\mathcal{C}(G)$ onto $\mathcal{C}(G//K)$ and its adjoint is $\mu \mapsto \mu^{\#}$:

$$\langle \mu, f^{\#} \rangle = \langle \mu^{\#}, f \rangle$$

further f is K-invariant if and only if $f = f^{\#}$ and μ is K-invariant if and only if $\mu = \mu^{\#}$.

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K-translations and K-varieties

Suppose that (G, K) is a Gelfand pair. Then the measures $\delta_y^\#$ commute for all y in G: for each f in $\mathcal{C}(G//K)$ we have

$$\langle \delta_y^\# * \delta_z^\#, f\rangle = \int_K f(ykz)\, d\omega(k) = \int_K f(zky)\, d\omega(k).$$

Similarly, the operators τ_y defined on $\mathcal{C}(G//K)$ by

$$\tau_y f = \delta_{y^{-1}}^\# * f = \int f(xz^{-1})\, d\delta_{y^{-1}}^\#(z) = \int_K f(xky)\, d\omega(k)$$

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K-spectral analysis

We say that K-spectral analysis holds for a K-variety if every nonzero K-subvariety of it contains a K-spherical function. We say that K-spectral analysis holds for G if K-spectral analysis holds for every K-variety.

In a commutative complex algebra A a maximal ideal M is called *exponential maximal ideal*, if A/M is isomorphic to the complex filed.

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K-spectral analysis holds for the K-variety V if and only if for every closed maximal ideal M of the residue algebra $\mathcal{M}_c(G//K)\Big/\mathrm{Ann}\,V$ is exponential. K-spectral analysis holds for G if and only if every closed maximal ideal of $\mathcal{M}_c(G//K)$ is exponential.

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Modified differences and K-monomials

For each K-spherical function s we define the modified K-difference

$$\Delta_{s;y} = \delta^{\#}_{y^{-1}} - s(y)\delta_e,$$

and their products $\Delta_{s;y_1,y_2,\ldots,y_{k+1}} = \Pi_{j=1}^{k+1}\Delta_{s;y_j}$. Given the K-spherical function s the closed ideal generated by all modified differences $\Delta_{s;y}$ with y in G is an exponential maximal ideal, denoted by M_s. The K-invariant f is called an s-monomial if $\dim \tau_K(f) < \infty$ is and there is a natural number k such that

$$M_s^{k+1} \subseteq \mathrm{Ann}\, \tau_K(f)$$

where $\tau_K(f)$ denotes the K-variety generated by f. This is equivalent to the functional equation

$$\Delta_{s;y_1,y_2,\ldots,y_{k+1}} * f(x) = 0$$

for each $x, y_1, y_2, \ldots, y_{k+1}$ in G. If f is nonzero, then s is uniquely determined, and the smallest k with this property is called the degree of the s-monomial f.

K-spectral synthesis

For instance, s-monomials of degree 2 are of the form $cs + f$, where f is a K-invariant continuous solutions of the *K-sine equation*:

$$\int_K f(xky)\, d\omega(k) = f(x)s(y) + f(y)s(x).$$

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We say that the K-variety is *K-synthesizable* if all K-monomials span a dense subspace in the variety. We say that *K-spectral synthesis holds* for a K-variety, if every nonzero subvariety of it is K-synthesizable.

K-spectral synthesis implies K-spectral analysis.

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The K-variety V is synthesizable if and only if its annihilator is the intersection of those cofinite closed ideals of $\mathcal{M}_c(G//K)$ which contain it.

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If K is a normal subgroup and G/K is commutative, then all these concepts coincide with the corresponding spectral analysis and synthesis concepts on the locally compact Abelian group G/K. Obviously, this is the case if G itself is commutative.

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Semidirect products

Let N be a locally compact topological group and K is a compact group of automorphisms of N. We consider the *semidirect product* of K and N: $K \ltimes N$: it is $K \times N$ equipped with the operation

$$(k, n) \cdot (l, m) = (k \circ l, (k \cdot m)n),$$

where $\circ$ is the composition of the automorphisms k, l, $\cdot$ is the effect of the automorphisms on the elements of N, and juxtaposition is the group operation in N. It turns out that this operation defines a group structure on $K \times N$, where the identity is (id, e), with the identity automorphism id of N and the identity element e of n, and the inverse of (k, n) is $(k^{-1}, k^{-1} \cdot u^{-1})$. With the product topology $G = K \ltimes N$ is a locally compact topological group, the *semidirect product* of K and N. The group N is topologically isomorphic to the closed normal subgroup $\{(id, n) : n \in N\}$, and the group K is topologically isomorphic to the compact subgroup $\{(k, e) : k \in K\}$. We shall identify these isomorphic groups:

$$K = \{(k, e) : k \in K\}, \;\; N = \{(id, n) : n \in N\}.$$

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Example: Affine groups

Let X be a finite dimensional vector space and K a compact subgroup of $GL(X)$, the *general linear group* of X with the normalized Haar measure ω. Then the set $K \times X$ acts on X: for S in K and u in V let $(S, u)x$ defined by the affine mapping

$$(S, u)x = Sx + u$$

for each x in V. The composition of affine mappings defines the operation on $K \times X$ as

$$(S, u) \cdot (T, v) = (S \circ T, Sv + u)$$

and with the identity $(id, 0)$ and inverse $(S, u)^{-1} = (S^{-1}, -S^{-1}u)$ we obtain the group

$$\mathrm{Aff}\, K = K \ltimes X,$$

the *semidirect product* of K and X. Here – as we have seen – K is topologically isomorphic to the compact subgroup $\{(S, 0) : S \in K\}$ and X is topologically isomorphic to the closed normal subgroup $\{(id, u) : u \in X\}$.

Example: Semidirect products

K-invariant functions are exactly those functions $(S, u) \mapsto f(S, u)$ which depend only on u and are invariant with respect to K:

$$f(S, u) = f(id, u) = f(id, Su)$$

for each S in K and u in X. Hence $\mathcal{C}(\mathrm{Aff}\, K//K)$ can be identified with a closed subspace of $\mathcal{C}(X)$, the space of K-*radial functions*. Similarly, the space of K-invariant measures $\mathcal{M}_c(\mathrm{Aff}\, K//K)$ on $\mathrm{Aff}\, K$ can be identified with a closed subspace of $\mathcal{M}_c(X)$, the space of K-*radial measures*,

Then $\mathrm{Aff}\, K$ is a locally compact group, K is topologically isomorphic to the compact subgroup $\{(L, 0) : L \in K\}$, and $\mathbb{R}^n$ is topologically isomorphic to the normal subgroup $\{(id, u) : u \in \mathbb{R}^n\}$.

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Example: The Poincaré group

The Poincaré group

We consider the real vector space $\mathbb{R}^{1,3} = \mathbb{R} \oplus \mathbb{R}^3$ equipped with the *indefinite inner product*

$$\langle v, w \rangle = v_0 w_0 - \sum_{j=1}^{3} v_j w_j,$$

where $v = (v_0, v_1, v_2, v_3)$ and $w = (w_0, w_1, w_2, w_3)$. The *isometry group* $O(1,3)$ of this indefinite inner product space is called the *Lorentz group*. The affine group of the Lorentz group

$$\mathrm{Aff}\, O(1,3) = O(1,3) \ltimes \mathbb{R}^{1,3}$$

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Example: Euclidean motions

The group of Euclidean motions

We consider the vector space $\mathbb{R}^n$ and the *orthogonal group* $O(n)$, the group of rotations. Together with translations they generate the group of *Euclidean motions*: rigid motions leaving the origin fixed. This is the affine group of $O(n)$:

$$\operatorname{Aff} O(n) = O(n) \ltimes \mathbb{R}^n$$

which acts on $\mathbb{R}^n$ by

$$(O, u)x = Ox + u$$

for O in $O(n)$ and x, u in $\mathbb{R}^n$.

Clearly, for $n = 1$ we have $O(1) = \{+1, -1\}$. $O(1)$-spherical functions are the functions of the form $x \mapsto \cosh \lambda x$ with arbitrary complex λ.

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Example: Proper Euclidean motions

The group of proper motions

We consider the vector space $\mathbb{R}^n$ and the *special orthogonal group* $SO(n)$, the group of proper rotations: orthogonal operators with determinant $+1$. Together with translations they generate the group of *proper Euclidean motions*: rigid motions which preserve orientation: no reflection is included. This is the affine group of $SO(n)$:

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Example: Proper Euclidean motions

The $SO(n)$-invariant functions can be identified with those continuous functions $f:\mathbb{R}^n\to\mathbb{C}$ with $f(Sx)=f(x)$ for each S in $SO(n)$ and x in $\mathbb{R}^n$. These are called *radial functions* as $f(x)$ depends only on $\|x\|$: $f(x)=\varphi(\|x\|)$ for some continuous $\varphi:\mathbb{R}\to\mathbb{C}$. Similarly, $\mathcal{M}_c(\mathrm{Aff}\,SO(n))$ is identified with those measures in $\mathcal{M}_c(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} f(Sx)\,d\mu(x)=\int_{\mathbb{R}^n} f(x)\,d\mu(x)$$

for each f in $\mathcal{C}(\mathbb{R}^n)$ and S in $SO(n)$: *radial measures*.

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Radial functions: $\mathcal{C}_r(\mathbb{R}^n) \approx \mathcal{C}(\mathrm{Aff}\,(SO(n))//SO(n))$

Radial measures: $\mathcal{M}_r(\mathbb{R}^n) \approx \mathcal{M}_c(\mathrm{Aff}\,(SO(n))//SO(n))$

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Example: Proper Euclidean motions

$SO(n)$-**translation**: for f in $\mathcal{C}_r(\mathbb{R}^n)$ and y in $\mathbb{R}^n$

$$\tau_y(f)(x) = \int_{SO(n)} f(x+ky)\,d\omega(k)$$

$SO(n)$-**variety**: $V \subseteq \mathcal{C}_r(\mathbb{R}^n)$ linear subspace, closed with respect to uniform convergence on compact sets, and for each f in V and y in $\mathbb{R}^n$ we have $x \mapsto \int_{SO(n)} f(x+ky)\,d\omega(k)$ is in V

$SO(n)$-**spherical function**: $s \neq 0$ in $\mathcal{C}_r(\mathbb{R}^n)$ and

$$\int_{SO(n)} s(x+ky)\,d\omega(k) = s(x)s(y) \text{ for each } y \in \mathbb{R}^n$$

Example: Proper Euclidean motions

Eigenfunctions of the Laplacian

The $SO(n)$-spherical functions are exactly the normalized radial eigenfunctions of the Laplacian in $\mathbb{R}^n$.

Let

$$\varphi(\|x\|) = s(x) \text{ for } x \in \mathbb{R}^n,$$

then, using the radial form of the Laplacian in $\mathbb{R}^n$ we have the Bessel differential equation

$$\frac{d^2}{dr^2}\varphi(r) + \frac{n-1}{r}\frac{d}{dr}\varphi(r) = \lambda\varphi(r),$$

with φ is regular at 0 and $\varphi(0) = 1$. Let J_λ denote the function

$$J_\lambda(r) = \Gamma\Big(\frac{n}{2}\Big) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\,\Gamma(k+\frac{n}{2})} \Big(\frac{r}{2}\Big)^{2k}.$$

Then s is an $SO(n)$-spherical function if and only if

$$s(x) = s_\lambda(x) = J_\lambda(\|x\|)$$

holds for each x in $\mathbb{R}^n$ with some complex number λ.

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$$J_\lambda(r) = \Gamma\Big(\frac{n}{2}\Big) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\,\Gamma(k+\frac{n}{2})} \Big(\frac{r}{2}\Big)^{2k}.$$

Then s is an $SO(n)$-spherical function if and only if

$$s(x) = s_\lambda(x) = J_\lambda(\|x\|)$$

holds for each x in $\mathbb{R}^n$ with some complex number λ.

$SO(n)$-monomials

Derivatives with respect to the parameter

Given the $SO(n)$-spherical function s_λ with some complex λ the s_λ-monomials of degree at most k are exactly the linear combinations of the derivatives $\frac{d^j}{d\lambda^j} s_\lambda$ for $j = 0, 1, \ldots, k$.

$SO(n)$-spectral analysis and synthesis

Every nonzero variety contains an $SO(n)$-spherical function, moreover, all functions of the form $\frac{d^j}{d\lambda^j} s_\lambda$ span a dense subspace in every variety.

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As $SO(1) = \{id\}$, hence $\mathrm{Aff}\, SO(1) = \mathbb{R}$, $SO(1)$-varieties are exactly the closed translation invariant subspaces of $\mathcal{C}(\mathbb{R})$. $SO(1)$-spherical functions are exactly the exponentials: $s_\lambda(x) = e^{\lambda x}$, and $SO(1)$-monomials are the linear combinations of the functions

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