Approximation by Poisson polynomials in Smirnov classes with variable exponent

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The variable exponent Lebesgue spaces $L^{p(x)}$ are a generalization of the classical Lebesgue spaces L^p, when the constant exponent p replace by a exponent function $p(\cdot)$. Lebesgue spaces with variable exponent provide us further advantages.

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at this time we have

$$\int_{\mathbb{R}} |f(x)|^{p(x)}\, dx < \infty.$$

More detail can be found in monograph [Cruz-Uribe, Fiorenza 2013].

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This close ties related with mechanics and physics make Lebesgue spaces with variable exponent important in time. Accordingly investigating some properties of these function space gains acceleration.

The fundamental problems of the approximation theory in the variable exponent Lebesgue spaces of periodic and non periodic functions defined on the intervals of real line were studied and solved by different authors. The detailed information about these spaces can be found in the monographs : [Sharapudinov 2012] and [Cruz-Uribe, Fiorenza 2013].

Introduction

In this talk we are going to mention that approximation properties of Poisson polynomials in addition to this direct and inverse theorems of approximation theory in Smirnov classes with variable exponent.

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Let also $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D} := Int\ \mathbb{T}$ and $\mathbb{D}^{-} := Ext\ \mathbb{T}$.

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ for a given Lebesgue measurable variable exponent $p(z) \geq 1$ on Γ we define as the set of Lebesgue measurable functions f, such that

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$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf \left\{ \lambda > 0 : \int\limits_\Gamma |f(z)/\lambda|^{p(z)} |dz| \leq 1 \right\} < \infty.$$

Smirnov Classes With Variable Exponent

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If $p(\cdot) = p$ it is the classical Lebesgue space $L^{p}(\Gamma)$.

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$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\left\{\lambda > 0 : \int_0^{2\pi} \left|f(e^{it})/\lambda\right|^{p(e^{it})} |dt| \leq 1\right\} =: \|f\|_{L^{p(\cdot)}([0,2\pi])}\,.$$

Let f be an analytic function in region G. If there exists a sequence of rectifiable Jordan curves (γ_n) in G, tending to the boundary Γ such that

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Each function $f \in E^p(G)$ has [Goluzin 1969, pp. 419-438] the non-tangential boundary values almost everywhere $(a.e)$ on Γ and the boundary function belongs to $L^p(\Gamma)$.

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$$\|f\|_{E^{p(\cdot)}(G)} := \|f\|_{L^{p(\cdot)}(\Gamma)},$$

$E^{p(\cdot)}(G)$ becomes the Banach spaces.

$\mathcal{P}(\cdot)$ Condition

Let $\mathcal{E}$ be the segment $[0, 2\pi]$ or a Jordan rectifiable curve Γ and let $p(\cdot) : \mathcal{E} \to \mathbb{R}^{+} := [0, \infty)$ be a Lebesgue measurable function defined on $\mathcal{E}$ such that

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$$|p(z_1) - p(z_2)| \ln \left(\frac{|\mathcal{E}|}{|z_1 - z_2|} \right) \leq c, \quad \forall z_1, z_2 \in \mathcal{E},$$

with a positive constant $c(p)$, *where* $|\mathcal{E}|$ *is the Lebesgue measure of* $\mathcal{E}$.

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with a positive constant $c(p)$, *where* $|\mathcal{E}|$ *is the Lebesgue measure of* $\mathcal{E}$. *If* $p(\cdot) \in \mathcal{P}(\mathcal{E})$ *and* $p_- > 1$, *then we say that* $p(\cdot) \in \mathcal{P}_0(\mathcal{E})$.

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Definition 2 *If the curve* Γ *has a parametrization*

$$\Gamma:g\left(t\right)\ ,\quad 0\leq t\leq 2\pi$$

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$$\int_0^{2\pi} \frac{\omega(g', t)}{t} dt < \infty,$$

then Γ *is called Dini smooth curve* [Pommerenke 1992, p. 48].

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The set of Dini smooth curves is denoted by $\mathfrak{D}$ in this talk.

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Conformal Maps

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Let ψ abe the inverse mapping of φ. The φ and ψ have continuous extensions to Γ and $\mathbb{T}$,respectively.Their derivatives $\varphi^{'}$ and ψ' have definite nontangential boundary values *a.e.* on Γ and $\mathbb{T}$, and the boundary functions are integrable with respect to Lebesgue measure on Γ and $\mathbb{T}$, respectively [Goluzin 1969, p. 419-438].

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$$f \in L^{p(\cdot)}(\Gamma) \Leftrightarrow f_0 \in L^{p_0(\cdot)}(\mathbb{T}) \text{ and } p_0 \in \mathcal{P}(\mathbb{T}) \Leftrightarrow p \in \mathcal{P}(\Gamma).$$

Cauchy Singular Operator on Γ

Let

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int\limits_{\Gamma\setminus\{t\in\Gamma:|t-z|<\varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

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be the Cauchy singular operator in $L^{p(\cdot)}(\Gamma)$. By Privalov's lemma the Cauchy type integrals

$$f^{+}(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G$$

$$f^{-}(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G^{-}$$

have the nontangential inside and outside limits *a.e.* on Γ respectively.

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Boundedness Cauchy Singular Operator

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Theorem A

Let $\Gamma \in \mathfrak{D}$ *and* $p \in \mathcal{P}_0(\Gamma)$. *If* $f \in L^{p(\cdot)}(\Gamma)$ *then Cauchy singular operator* $S_\Gamma(f)$ *is bounded operator in* $L^{p(\cdot)}(\Gamma)$.

For a given function $f \in L^{p(\cdot)}(\Gamma)$ we define the Cauchy type integral

$$f_0^+(w) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau, \quad w \in \mathbb{D}$$

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Lemma 1 [Israfilov, Testici 2015]

If $f \in L^{p(\cdot)}(\Gamma)$, $\Gamma \in \mathfrak{D}$, *and* $p \in \mathcal{P}_0(\Gamma)$ *then*

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Lemma 1 [Israfilov, Testici 2015]

If $f \in L^{p(\cdot)}(\Gamma)$, $\Gamma \in \mathfrak{D}$, *and* $p \in \mathcal{P}_0(\Gamma)$ *then* $f^+ \in E^{p(\cdot)}(G)$ and $f^- \in E^{p(\cdot)}(G^-)$.

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Lemma 1 [Israfilov, Testici 2015]

If $f \in L^{p(\cdot)}(\Gamma)$, $\Gamma \in \mathfrak{D}$, and $p \in \mathcal{P}_0(\Gamma)$ then $f^+ \in E^{p(\cdot)}(G)$ and *$f^- \in E^{p(\cdot)}(G^-)$.*

Therefore we obtain that if $f \in E^{p(\cdot)}(G)$, $\Gamma \in \mathfrak{D}$, and $p \in \mathcal{P}_0(\Gamma)$ then $f_0^+ \in E^{p_0(\cdot)}(\mathbb{D})$ and $f_0^- \in E^{p_0(\cdot)}(\mathbb{D}^-)$.

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$$\Delta_t^r f(w) := \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} f\left(we^{ist}\right) \;\; ; \;\; w \in \mathbb{T}, \; r = 1, 2, 3, \ldots$$

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$$\Omega_r(f, \delta)_{\mathbb{T}, p(\cdot)} := \sup_{0 < |h| \le \delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f(w)\, dt \right\|_{L^{p(\cdot)}(\mathbb{T})} .$$

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We define the modulus of smoothness for $f \in E^{p(\cdot)}(G)$ as following:

$$\Omega_r(f, \delta)_{G, p(\cdot)} \; : \; = \Omega_r\left(f_0^+, \delta\right)_{\mathbb{T}, p_0(\cdot)}.$$

$F_k(z)$, $k = 1, 2, ...$ Faber polynomails for continuum $\overline{G}$ are Laurent coefficients in the following series expansion:

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \quad z \in G \text{ and } w \in \mathbb{D}^-.$$

Faber Polynomials

$F_k(z)$, $k = 1, 2, \ldots$ Faber polynomails for continuum $\overline{G}$ are Laurent coefficients in the following series expansion:

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If $f \in E^{p(\cdot)}(G)$ then

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z)$$

where $z \in G$ and

$$a_k = a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw.$$

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Remark

Let $\Gamma \in \mathfrak{D}$, and $p \in \mathcal{P}_0(\Gamma)$. By taking into account $f_0^+ \in E^{p_0(\cdot)}(\mathbb{D})$ and $f_0^- \in E^{p_0(\cdot)}(\mathbb{D}^-)$ we can conclude that Faber coefficient of function f are Taylor coefficients of the functions f_0^-.

$T(f)$ Operator

Let Π be the set of all algebraic polynomials (with no restriction on the degree) and let $\Pi(\mathbb{D})$ be set of traces of members of Π on $\mathbb{D}$. If we define the operator $T : \Pi(\mathbb{D}) \subset E^{p_0(\cdot)}(\mathbb{D}) \to E^{p(\cdot)}(G)$:

$$T(f)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w)\,\psi'(w)}{\psi(w) - z} dw, \quad z \in G \text{ and } f \in E^{p_0(\cdot)}(\mathbb{D}).$$

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Theorem B [Israfilov, Testici 2015]

Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. Then,the operator

$$T : E^{p_0(\cdot)}(\mathbb{D}) \to E^{p(\cdot)}(G)$$

is linear, bounded, one-to-one and onto. Moreover,

$$T(f_0^+) = f \textit{ forevery } f \in E^{p(\cdot)}(G).$$

Best Approximation Number

Let Π_n^* be the class of algebraic polynomials of degree not exceeding n. The *best approximation number* of $f \in L^{p(\cdot)}(\Gamma)$ is defined by

$$E_n(f)_{G,p(\cdot)} := \inf \left\{ \|f - P_n\|_{L^{p(\cdot)}(\Gamma)} : P_n \in \Pi_n^* \right\} \quad n = 0, 1, 2, ..$$

For $f \in L^{p(\cdot)}(\mathbb{T})$ we define the best approximation number

$$E_n(f)_{p(\cdot)} := \inf \left\{ \|f - T_n\|_{p(\cdot)} : T_n \in \Pi_n \right\} \quad n = 0, 1, 2, ..$$

in the class Π_n of the trigonometric polynomials of degree not exceeding n.

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Lemma 2 [Israfilov, Testici 2015]

Let $\Gamma \in \mathfrak{D}$ *and* $p(\cdot) \in \mathcal{P}_0(\Gamma)$.

Best Approximation Number

Let Π_n^* be the class of algebraic polynomials of degree not exceeding n. The *best approximation number* of $f \in L^{p(\cdot)}(\Gamma)$ is defined by

$$E_n(f)_{G,p(\cdot)} := \inf \left\{ \|f - P_n\|_{L^{p(\cdot)}(\Gamma)} : P_n \in \Pi_n^* \right\} \quad n = 0, 1, 2, ..$$

For $f \in L^{p(\cdot)}(\mathbb{T})$ we define the best approximation number

$$E_n(f)_{p(\cdot)} := \inf \left\{ \|f - T_n\|_{p(\cdot)} : T_n \in \Pi_n \right\} \quad n = 0, 1, 2, ..$$

in the class Π_n of the trigonometric polynomials of degree not exceeding n.

Lemma 2 [Israfilov, Testici 2015]

Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. If $f \in E^{p(\cdot)}(G)$ then there exist the positive constants such that

$$E_n\left(f_0^+\right)_{p_0(\cdot)} \leq c_5(p) E_n(f)_{G,p(\cdot)} \leq c_6(p) E_n\left(f_0^+\right)_{p_0(\cdot)}.$$

Let $f \in E^{p(\cdot)}(G)$. Let $F_k(z)$ be Faber polynomials for $\overline{G}$ and $a_k, k = 0, 1, 2, ...$ be Faber coefficents of f.

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$$V_n(f, z) := \sum_{k=0}^{n} a_k F_k(z) + \sum_{k=n+1}^{2n-1} \left(2 - \frac{k}{n}\right) a_k F_k(z) \quad , \ z \in G.$$

Main Results

The following direct and inverse theorems of approximation theory in Lebesgue space with variable exponent is proved in [Israfilov, Testici 2017].

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$$E_n(f)_{p(\cdot)} \leq c(p)\, \Omega_r(f, 1/n)_{\mathbb{T},p(\cdot)}\,.$$

Theorem 2 [Israfilov, Testici 2017]

Let $p(\cdot) \in \mathcal{P}(\mathbb{T})$. *Then there exists a positive constant* $c(p, r)$ *such that for every* $f \in L^{p(\cdot)}(\mathbb{T})$ *and* $n = 0, 1, 2, \ldots$

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is holds, with a positive constant $c(p)$ *independent of* n.

By *Theorem 1*, *Theorem B* and *Lemma 2* we obtain :

Theorem 4

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$$E_n(f)_{G,p(\cdot)} \leq c(p,r)\,\Omega_r(f,1/n)_{G,p(\cdot)} \ , \ n = 1,2,3,...$$

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