One-sided mean approximation on the Euclidean sphere to the characteristic function of a spherical layer by algebraic polynomials

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Notation. Statement of the problem

Let $\mathbb{R}^m$, $m \geq 2$, be the Euclidean space with the inner product

$$xy = \sum_{k=1}^{m} x_k y_k,$$

$$x = (x_1, x_2, \dots, x_m), \quad y = (y_1, y_2, \dots, y_m),$$

and the norm $|x| = \sqrt{xx}$.

For $r > 0$, let $\mathbb{S}^{m-1}(r) = \{x \in \mathbb{R}^m\colon |x| = r\}$ be the sphere of radius r centered at the origin

$\mathbb{S}^{m-1} = \mathbb{S}^{m-1}(1)$

For a pair of numbers

$$\eta = (a, b), \qquad -1 \le a < b \le 1,$$

consider the spherical layer

$$\mathbb{G}(\eta) = \{x = (x_1, x_2, \dots, x_m) \in \mathbb{S}^{m-1} \colon a \le x_m \le b\}$$

centered at the "north pole" $e_m = (0, 0, \dots, 0, 1)$ of the sphere.

In the case $b = 1$, $a = h$, $-1 < h < 1$, the set $\mathbb{G}(\eta)$ is the spherical cap

$$\mathbb{C}(h) = \{x = (x_1, x_2, \dots, x_m) \in \mathbb{S}^{m-1} \colon x_m \ge h\}.$$

Let $L(E) = L_1(E)$ be the space of functions measurable and integrable on a set E with the norm

$$\|f\|_{L(E)} = \int_E |f(x)|dx.$$

Let $L_\infty(E)$ be the space of measurable essentially bounded functions on E with the norm

$$\|f\|_{L_\infty(E)} = \operatorname{ess\,sup}\{|f(x)| \colon x \in E\};$$

this is the conjugate space for $L(E)$.

On the unit sphere $\mathbb{S}^{m-1}$ of the space $\mathbb{R}^m$, $m \geq 2$, consider the classical $(m-1)$-dimensional Lebesgue measure. For a subset $E \subset \mathbb{S}^{m-1}$, denote by $|E|_{m-1}$ or $|E|$ its measure.

Denote by $\mathcal{P}_{n,m}$ the set of algebraic polynomials

$$P_n(x) = \sum_{\substack{|\alpha| = \alpha_1 + \cdots + \alpha_m \le n, \\ \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m}} c_\alpha x^\alpha,$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}, \qquad x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m,$$

of degree (at most) n in m variables with real coefficients c_α.

For a pair of measurable functions f and g on the sphere $\mathbb{S}^{m-1}$, the inequality $f \le g$ means that $f(x) \le g(x)$ for almost all $x \in \mathbb{S}^{m-1}$.

For a function f measurable and bounded on the sphere $\mathbb{S}^{m-1}$, consider the sets

$$\mathcal{P}^{-}_{n,m}(f) = \{P \in \mathcal{P}_{n,m}\colon P \le f\},$$

$$\mathcal{P}^{+}_{n,m}(f) = \{P \in \mathcal{P}_{n,m}\colon P \ge f\}.$$

In order that these sets were nonempty, assume that f is bounded from below and from above, respectively.

Consider the values of the best approximation in the space L to a function f by the set $\mathcal{P}_{n,m}$ from below and from above:

$$e^{-}_{n,m}(f) = \text{inf}\{\|f - P\|\colon P \in \mathcal{P}^{-}_{n,m}(f)\},$$

$$e^{+}_{n,m}(f) = \text{inf}\{\|f - P\|\colon P \in \mathcal{P}^{+}_{n,m}(f)\}.$$

Polynomials at which the infima are attained are called the *polynomials of best* (*integral*) *approximation to the function f from below* and *from above*, respectively, or *extremal polynomials*.

The main aim of this study is the best approximation from below in the space $L(\mathbb{S}^{m-1})$ to the charateristic function

$$\mathbf{1}_{\mathbb{G}(\eta)}(x) = \begin{cases} 1, & x \in \mathbb{G}(\eta), \\ 0, & x \notin \mathbb{G}(\eta), \end{cases}$$

of the spherical layer $\mathbb{G}(\eta)$ by the set of polynomials $\mathcal{P}^{-}_{n,m}(\mathbf{1}_{\mathbb{G}(\eta)})$. More exactly, we study the value

$$\begin{gathered} e^{-}_{n,m}(\mathbf{1}_{\mathbb{G}(\eta)}) = \\ = \inf\{\|\mathbf{1}_{\mathbb{G}(\eta)} - P_n\|_{L(\mathbb{S}^{m-1})} \colon P_n \in \mathcal{P}^{-}_{n,m}(\mathbf{1}_{\mathbb{G}(\eta)})\}; \end{gathered} \tag{1}$$

here, according to the notation introduced above,

$$\mathcal{P}^{-}_{n,m}(\mathbf{1}_{\mathbb{G}(\eta)}) = \{P_n \in \mathcal{P}_{n,m} : P_n \leq \mathbf{1}_{\mathbb{G}(\eta)}\}.$$

The crucial fact is that the function $\mathbf{1}_{\mathbb{G}(\eta)}(x)$, $x = (x_1, x_2, \dots, x_m)$, defined on the sphere $\mathbb{S}^{m-1}$, is zonal; i.e., this functions depends only on the coordinate x_m of the point $x = (x_1, x_2, \dots, x_m) \in \mathbb{S}^{m-1}$:

$$f(x_1, x_2, \dots, x_m) = g(x_m),\ x = (x_1, x_2, \dots, x_m) \in \mathbb{S}^{m-1},$$

where g is a univariate function defined on the interval$[-1, 1]$.

For the function $f = \mathbf{1}_{\mathbb{G}(\eta)}$, the function g in this relation is the characteristic function of the interval $[a, b]$.

Reduction to a one-dimensional problem

The passage to polar coordinates on the sphere $\mathbb{S}^{m-1}$ leads to the following representation of the integral of a function $f \in L(\mathbb{S}^{m-1})$ over the unit sphere:

$$\int\limits_{\mathbb{S}^{m-1}} f(x)dx = |\mathbb{S}^{m-2}| \int\limits_{-1}^{1} g(t)(1-t^2)^{(m-3)/2}\, dt\,.$$

where

$$g(t) = \frac{1}{\left|\mathbb{S}^{m-2}\right|} \int\limits_{\mathbb{S}^{m-2}} f\left(\sqrt{1-t^2}\, x', t\right) dx'.$$

For a real number t, denote by $\Lambda(t)$ the hyperplane of points $x = (x_1, x_2, \dots, x_{m-1}, t) \in \mathbb{R}^m$. We will write points $x = (x_1, x_2, \dots, x_{m-1}, t) \in \Lambda(t)$ in the form $x = (x_1, x_2, \dots, x_{m-1}, t) = (x', t),\ \ x' = (x_1, x_2, \dots, x_{m-1}) \in \mathbb{R}^{m-1}$.

For $t \in (-1, 1)$, the section of the sphere $\mathbb{S}^{m-1}$ by the hyperplane $\Lambda(t)$ is the $(m-2)$-dimensional sphere $\mathbb{S}^{m-2}(a)$ of radius $a = a(t) = \sqrt{1 - t^2}$ centered at the point te_m and parallel to the coordinate space $\mathbb{R}^{m-1}$ of points $x' = (x_1, x_2, \dots, x_{m-1})$. We identify this sphere with the sphere $\mathbb{S}^{m-2}(a) \subset \mathbb{R}^{m-1}$.

The function

$$g(t) = \frac{1}{\left|\mathbb{S}^{m-2}\right|} \int_{\mathbb{S}^{m-2}} f\left(\sqrt{1-t^2}\, x', t\right) dx'$$

can be interpreted as the averaging $g = Sf$ of the function f over sections of the sphere by hyperplanes. The averaging operator S defined by this formula is a bounded linear operator from the space $L(\mathbb{S}^{m-1})$ to the space $L_1^{\phi}(-1,1)$ of functions integrable over the interval $(-1,1)$ with the ultraspherical weight

$$\phi(t) = (1-t^2)^{\alpha}, \qquad \alpha = \frac{m-3}{2}.$$

For the averaging operator, we have the inequality

$$|\mathbb{S}^{m-2}| \cdot \|Sf\|_{L_1^{\phi}(-1,1)} \leq \|f\|_{L(\mathbb{S}^{m-1})}, \quad f \in L(\mathbb{S}^{m-1}).$$

For an algebraic polynomial $P_n \in \mathcal{P}_{n,m}$ of degree n in m variables, the function

$$g_n(t) = (SP_n)(t) = \frac{1}{\left|\mathbb{S}^{m-2}\right|} \int_{\mathbb{S}^{m-2}} P_n\left(\sqrt{1-t^2}\, x', t\right) dx'$$

is a univariate algebraic polynomial of the same degree n.

Thus, $S\mathcal{P}_{n,m} \subset \mathcal{P}_n,\ \mathcal{P}_n = \mathcal{P}_{n,1}$.

Actually, it is not hard to understand that

$$S\mathcal{P}_{n,m} = \mathcal{P}_n.$$

Lemma 1. *Let* $m \geq 3$. *If a function* f *is defined, integrable, bounded on the sphere* $\mathbb{S}^{m-1}$, *and zonal, then*

$$S(\mathcal{P}_{n,m}^{-}(f)) = \mathcal{P}_{n}^{-}(Sf).$$

This fact is quite obvious.

The function $\mathbf{1}_{\mathbb{G}(\eta)}$ is zonal; more exactly,

$$\mathbf{1}_{\mathbb{G}(\eta)}(x) = \mathbf{1}_{\mathbb{I}(\eta)}(x_m), \quad x = (x_1, x_2, \ldots, x_m) \in \mathbb{S}^{(m-1)},$$

where $\mathbf{1}_{\mathbb{I}(\eta)}$ is the characteristic function of the interval $\mathbb{I} = \mathbb{I}(\eta) = (a, b)$:

$$\mathbf{1}_{\mathbb{I}(\eta)}(t) = \begin{cases} 1, & t \in (a, b), \\ 0, & t \in [-1, 1] \setminus (a, b). \end{cases}$$

Consider the best approximation from below

$$\begin{gathered} E^{-}_{n,\phi}(\mathbf{1}_{\mathbb{I}(\eta)}) = \\ = \inf\{\|\mathbf{1}_{\mathbb{I}(\eta)} - p_n\|_{L_1^{\phi}(-1,1)} : p_n \in \mathcal{P}^{-}_n(\mathbf{1}_{\mathbb{I}(\eta)})\} \end{gathered} \tag{2}$$

to the step function $\mathbf{1}_{\mathbb{I}(\eta)}$ in the space $L_1^{\phi}(-1, 1)$ by the set $\mathcal{P}^{-}_n(\mathbf{1}_{\mathbb{I}(\eta)}) = \mathcal{P}^{-}_{n,1}(\mathbf{1}_{\mathbb{I}(\eta)})$ of (univariate) algebraic polynomials whose graphs lie under the graph of the function $\mathbf{1}_{\mathbb{I}(\eta)}$.

Lemma 2. *For any* $m \geq 3$, $n \geq 0$, *and* $a, b \in (-1, 1)$, *we have*

$$e^{-}_{n,m}(\mathbf{1}_{\mathbb{G}(\eta)}) = |\mathbb{S}^{m-2}|\, E^{-}_{n,\phi}(\mathbf{1}_{\mathbb{I}(\eta)})$$

and if a polynomial p^*_n *in one variable is extremal in problem* (2) (*i.e., the infimum in* (2) *is attained at this polynomial*), *then the zonal polynomial* $P^*_n(x) = p^*_n(x_m)$, $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, *is extremal in problem* (1) *on the sphere.*

One-sided approximation on an interval

Consider in more detail the problem of one-sided approximation from below to the characteristic function

$$\mathbf{1}_{\mathbb{I}}(t) = \begin{cases} 1, & t \in (a,b), \\ 0, & t \notin (a,b), \end{cases}$$

of the interval $\mathbb{I} = (a,b)$ by the set of algebraic polynomials (in one variable) of given degree $n \geq 0$ in the space $L^{\psi} = L^{\psi}(-1,1)$ with a more general nonnegative weight ψ (as compared to the ultraspherical weight ϕ) on $(-1,1)$. The problem is in calculating the value

$$E^{-}_{n,\psi}(\mathbf{1}_{\mathbb{I}}) = \inf\{\|\mathbf{1}_{\mathbb{I}} - p_n\|_{L_1^{\psi}(-1,1)} : p_n \in \mathcal{P}^{-}_{n}(\mathbf{1}_{\mathbb{I}})\} \qquad (3)$$

[BMQ] **Bustamante J., Martínez Cruz R., Quesada J.M.** Quasi orthogonal Jacobi polynomials and best one-sided L_1 approximation to step functions, J. Approx. Theory. 2015. Vol. 198. P. 10–23.

[BDR] **Babenko A.G., Deikalova M.V., Révész Sz.G.**
Weighted one-sided integral approximations to characteristic functions of intervals by polynomials on a closed interval, Proc. Steklov Inst. Math. 2017. Vol. 297, Suppl. 1. P. S11–S18

In these papers, problem (3) was solved in the case when one of the end-points of the interval $\mathbb{I} = (a, b)$ coincides with the corresponding end-point ± 1 of the initial interval $(-1, 1)$.

These results and the method of their proving allow us to solve problem (3) under the assumption that the weight ψ is even and the interval $\mathbb{I} = (a, b)$ is symmetric about 0; i.e., $a = -h$ and $b = h$, where $0 < h < 1$.

Indeed, consider the auxiliary problem of approximation of the function

$$\mathbf{1}_h(t) = \begin{cases} 1, & t \in [0, h^2), \\ 0, & t \in [h^2, 1], \end{cases}$$

on the interval $[0, 1]$. Studying this problem, we use the results and method of [BDR] applied to the interval $[0, 1]$.

To obtain lower bounds, the authors of [BDR] followed the known scheme and used quadrature formulas with positive coefficients. Similarly, consider the positive quadrature formula

$$\int_0^1 p(u)\psi(u)\,du = \sum_{k=1}^{M} \lambda_k p(u_k), \qquad p \in \mathcal{P}_\ell, \qquad (4)$$

which is valid on the set $\mathcal{P}_\ell$ of polynomials of degree ℓ, h^2 is its node, and all nodes $\{u_k\}$ lie inside the interval $[0, 1]$. The extremal polynomial p^* in the approximation problem interpolates the function $\mathbf{1}_h$ at the nodes of quadrature formula (4). The method of constructing the polynomial goes back to A.Markov (1883) and Stieltjes (1884).

Making the change of variable $u = t^2$ on the left-hand side of (4) and considering the polynomial $q(t) = p(t^2)$, we obtain

$$2 \int_0^1 q(t)\psi(t^2)t\,dt = \sum_{k=1}^{M} \lambda_k q(\sqrt{u_k}). \tag{5}$$

Making the change of variable $v = -t$ in (5), we obtain

$$2 \int_{-1}^{0} q(v)\psi(v^2)|v|\,dv = \sum_{k=1}^{M} \lambda_k q(-\sqrt{u_k}). \tag{6}$$

Combining (5) and (6), we obtain the new quadrature formula on the interval $[-1, 1]$:

$$\int_{-1}^{1} q(t)\psi(t^2)|t|\,dt = \sum_{k=1}^{M} \frac{\lambda_k}{2}(q(-\sqrt{u_k}) + q(\sqrt{u_k})). \qquad (7)$$

This quadrature formula is valid for even polynomials q of degree 2ℓ. For odd polynomials q of any degree, formula (7) is also valid; moreover, its left- and right-hand sides are zero.

Thus, quadrature formula (7) is valid for polynomials of degree $2\ell + 1$, its coefficients are positive, and the points $-h$ and h are its nodes. This quadrature formula is extremal in problem (2) with the weight $\psi(t^2)|t|$. The extremal polynomial in this problem is $q^*(t) = p^*(t^2)$, which interpolates the function $\mathbf{1}_{\mathbb{I}(\eta)}$ at the nodes of quadrature formula (7). The value of the best approximation is

$$\int_{-h}^{h} \psi(t^2)|t|\,dt - \sum_{k=1}^{S} \lambda_k, \qquad u_S = h^2.$$

One-sided approximation of the characteristic function $\mathbf{1}_{\mathbb{I}}$ of the interval $\mathbb{I} = (a, b)$ for an arbitrary pair of points a and b such that $-1 < a < b < 1$ turned to be a difficult problem. We know its solution only in some cases:

(i) In the case when the points a and b are neighboring nodes of a positive quadrature formula valid on the set of polynomials of certain order n. In this case, the polynomial $p_n^* \equiv 0$ is the best approximation from below.

(ii) For polynomials of small degree n and specific points a and b; for example, for $n = 7$, the unit weight, and in the case when a and b are an arbitrary pair of the Gauss four-point quadrature formula (which is valid for polynomials of degree 7). In this case, the extremal polynomial interpolates the function $\mathbf{1}_{\mathbb{I}}$ at the nodes of the Gauss quadrature formula.

One-sided mean approximation to the characteristic function of a spherical layer by algebraic polynomials

For the present, we know a solution of the initial problem (1) of the one-sided approximation from below in the space $L(\mathbb{S}^{m-1})$ to the characteristic function of the spherical layer

$$\mathbb{G}(\eta) = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{S}^{m-1} \colon a \le x_m \le b\},$$

$\eta = (a, b)$, $-1 \le a < b \le 1$, by algebraic polynomials of arbitrary degree in m variables in the following cases:

(1) for $b = 1$ and $a = h$, where $-1 < h < 1$, when the layer becomes the spherical cap

$$\mathbb{C}(h) = \{x = (x_1, x_2, \dots, x_m) \in \mathbb{S}^{m-1} \colon x_m \geq h\};$$

(2) for a layer symmetrical about the equator; more precisely, in the case $a = -h$ and $b = h$, where $0 < h < 1$.

(3) The points a and b are neighboring nodes of a positive quadrature formula valid on the set of polynomials of certain order n.

(4) In the three-dimensional space ($m = 3$) for polynomials of small degree n and specific points a and b; for example, for $n = 7$, in the case when a and b are an arbitrary pair of the Gauss four-point quadrature formula (which is valid for polynomials of degree 7).

Thank you for your attention!