Convergence of rectangular summability and Lebesgue points of higher dimensional Fourier transforms

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Wiener amalgam spaces

The space $L_p(\mathbb{R}^d)$ is equipped with the norm

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^d} |f|^p \, d\lambda \right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{R}^d} |f|, & p = \infty. \end{cases}$$

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \leq p, q \leq \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k \in \mathbb{Z}^d} \|f(\cdot + k)\|_{L_p[0,1)^d}^q \right)^{1/q} < \infty,$$

with the obvious modification for $q = \infty$.
It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \leq p \leq \infty).$$

The θ-summability

The *Fourier transform* of $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \int_{\mathbb{R}} f(u) e^{-2\pi \imath x u} \, du \qquad (x \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \le p \le 2$, then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi \imath x t} \, du \qquad (x \in \mathbb{R}, \widehat{f} \in L_1(\mathbb{R})).$$

This motivates the definition of the Dirichlet integral $s_T f$:

$$s_T f(x) \quad := \quad \int_{-T}^{T} \widehat{f}(t) e^{2\pi \imath x t} \, dt.$$

Theorem (Carleson, Hunt, 1967)

For $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T \to \infty} s_T f = f \qquad a.e.$$

This convergence does not hold for $p = 1$. However, using a summability method, we can generalize these results. In the one-dimensional case let

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}} \theta\left(\frac{|t|}{T}\right) \widehat{f}(t) e^{2\pi\imath xt} \, du.$$

In the higher dimensional case,

$$\widehat{f}(x) = \int_{\mathbb{R}^d} f(u) e^{2\pi\imath x\cdot u} \, du \qquad (x \in \mathbb{R}^d)$$

and for $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$,

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(t) e^{2\pi\imath x\cdot t} \, dt \qquad (x \in \mathbb{R}^d, \widehat{f} \in L_1(\mathbb{R}^d)).$$

Let

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) \widehat{f}(t) e^{2\pi \imath x\cdot t}\, dt \qquad (T>0),$$

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^d \theta_j\left(\frac{|t_j|}{T_j}\right)\right) \widehat{f}(t) e^{2\pi \imath x\cdot t}\, dt \quad (T\in\mathbb{R}_+^d).$$

Suppose that $\theta(0)=\theta_i(0)=1$. This summation contains all well known summability methods, such as the Fejér, Riesz, Weierstrass, Abel, Picard, Bessel summations.

One-dimensional summability

By Lebesgue's theorem,

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(x-s)\, ds = f(x)$$

for a.e. $x \in \mathbb{R}$. Then

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(x-s) - f(x)\, ds = 0.$$

A point $x \in \mathbb{R}$ is called a *Lebesgue point* of $f \in W(L_1, \ell_\infty)(\mathbb{R})$ if

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} |f(x-s) - f(x)|\, ds = 0.$$

Theorem

Almost every point $x \in \mathbb{R}$ is a Lebesgue point of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

For $\theta(t) = \max((1-|t|),0)$ we obtain the Fejér means:

$$\begin{aligned}\sigma_T f(x) &:= \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) \widehat{f}(t) e^{2\pi \imath x t}\, dt \\ &= \frac{1}{T} \int_0^T s_t f(x)\, dt.\end{aligned}$$

Theorem (Lebesgue, Fejér, 1905)

For all Lebesgue points of $f \in L_1(\mathbb{R})$

$$\lim_{T\to\infty} \sigma_T f(x) = f(x).$$

Unrestricted rectangular summability

Recall that

$$\|f\|_{W(L_p,\ell_\infty)} := \left(\sup_{n\in\mathbb{Z}^d} \int_{n_1}^{n_1+1} \cdots \int_{n_d}^{n_d+1} |f(x)|^p \, dx \right)^{1/p}.$$

The norms of the iterated Wiener amalgam spaces $W_I(L_p,\ell_\infty)(\mathbb{R}^d)$ and $W_I(L_p(\log L)^{d-1},\ell_\infty)(\mathbb{R}^d)$ $(1 \le p \le \infty)$ are given by

$$\|f\|_{W_I(L_p,\ell_\infty)} := \left(\sup_{n_1\in\mathbb{Z}} \int_{n_1}^{n_1+1} \cdots \sup_{n_d\in\mathbb{Z}} \int_{n_d}^{n_d+1} |f(x)|^p \, dx_d \cdots dx_1 \right)^{1/p}$$

and

$$\|f\|_{W_I(L_p(\log L)^{d-1},\ell_\infty)}$$

$$:= \left(\sup_{n_1\in\mathbb{Z}} \int_{n_1}^{n_1+1} \cdots \sup_{n_d\in\mathbb{Z}} \int_{n_d}^{n_d+1} |f(x)|^p (\log^+ |f(x)|)^{d-1} \, dx_d \cdots dx_1 \right)^{1/p}.$$

A function f is in the set $L_p(\log L)^{d-1}(\mathbb{R}^d)$ $(1 \leq p < \infty)$ if

$$\|f\|_{L_p(\log L)^{d-1}} := \left(\int_{\mathbb{R}^d} |f(x)|^p (\log^+ |f(x)|)^{d-1} \, dx \right)^{1/p} < \infty.$$

Then

$$\begin{aligned} W(L_p, \ell_\infty)(\mathbb{R}^d) &\supset W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d) \\ &\supset L_p(\log L)^{d-1}(\mathbb{R}^d), C_0(\mathbb{R}^d), L_r(\mathbb{R}^d) \end{aligned}$$

for all $1 \leq p < r \leq \infty$.

We denote by $I(c,h)$ $(c \in \mathbb{R}, h > 0)$ the interval $\{x \in \mathbb{R} : |x-c| < h\}$. Let the dyadic coronas be defined by

$$Q_k := I(0,2^k) \setminus I(0,2^{k-1}) \quad (k > 0), \qquad Q_0 := I(0,1).$$

The Herz space $E_q(\mathbb{R})$ contains all functions f for which

$$\|f\|_{E_q} := \sum_{k=0}^{\infty} 2^{k(1-1/q)} \|f \mathbf{1}_{Q_k}\|_q < \infty.$$

Then

$$L_1(\mathbb{R}) = E_1(\mathbb{R}) \supset E_q(\mathbb{R}) \supset E_{q'}(\mathbb{R}) \supset E_\infty(\mathbb{R}), \qquad 1 < q < q' < \infty.$$

Theorem

If $f \in W_l(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$, *then*

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-s)\, ds = f(x) \qquad a.e.\ x \in \mathbb{R}^d.$$

Then

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-s) - f(x)\, ds = 0.$$

A point $x \in \mathbb{R}^d$ is called a strong *p-Lebesgue point* of f $(1 \leq p < \infty)$ if

$$\lim_{h \to 0} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-s) - f(x)|^p\, ds \right)^{1/p} = 0.$$

Here $h \to 0$ means that $h_1 \to 0, \ldots, h_d \to 0$.

Theorem

Almost every point $x \in \mathbb{R}^d$ *is a strong p-Lebesgue point of* $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$ *if* $1 \le p < \infty$.

Theorem (Weisz, 2014)

Let $\theta_i \in L_1(\mathbb{R})$, $1 \le p < \infty$ *and* $1/p + 1/q = 1$. *If* $\widehat{\theta}_i \in E_q(\mathbb{R})$ $(i = 1, \ldots, d)$, *then*

$$\lim_{T \to \infty} \sigma_T^\theta f(x) = f(x)$$

for all strong p-Lebesgue points of $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$.

Here $T \to \infty$ means that $T_1 \to \infty, \ldots, T_d \to \infty$.

Restricted rectangular summability over a cone

Suppose that $\theta_i \in L_1(\mathbb{R})$, $\widehat{\theta}_i \in L_1(\mathbb{R})$ $(i = 1, \ldots, d)$, $\delta \geq 1$ and

$$T = (T_1, \ldots, T_d) \in \mathbb{R}^d_\delta := \{T \in \mathbb{R}^d_+ : \delta^{-1} \leq T_i/T_j \leq \delta, i, j = 1, \ldots, d\}.$$

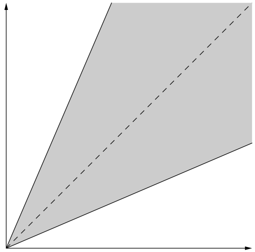

Figure: The cone for $d = 2$.

The weighted Herz space $E_q^\mu(\mathbb{R})$ contains all functions f for which

$$\|f\|_{E_q^\mu} := \sum_{k=0}^{\infty} 2^{k(\mu+1-1/q)} \|f \mathbf{1}_{Q_k}\|_q < \infty,$$

where

$$Q_k := I(0,2^k) \setminus I(0,2^{k-1}) \quad (k>0), \qquad Q_0 := I(0,1).$$

Then

$$E_q(\mathbb{R}) = E_q^0(\mathbb{R}) \supset E_q^\mu(\mathbb{R}) \qquad 0 \le \mu < \infty$$

and

$$L_1(\mathbb{R}) \supset E_1^\mu(\mathbb{R}) \supset E_q^\mu(\mathbb{R}) \supset E_{q'}^\mu(\mathbb{R}) \supset E_\infty^\mu(\mathbb{R}), \qquad 1 < q < q' < \infty.$$

Let $|i| := |i_1| + \cdots |i_d|$.

Recall the definition of the strong Lebesgue points:

$$\lim_{h\to 0}\left(\frac{1}{\prod_{j=1}^{d}(2h_j)}\int_{-h_1}^{h_1}\cdots\int_{-h_d}^{h_d}|f(x-s)-f(x)|^p\,ds\right)^{1/p}=0.$$

The definition of the strong Lebesgue points is equivalent to

$$\lim_{r\to 0}\sup_{h_j<r,j=1\ldots,d}\left(\frac{1}{\prod_{j=1}^{d}(2h_j)}\int_{-h_1}^{h_1}\cdots\int_{-h_d}^{h_d}|f(x-s)-f(x)|^p\,ds\right)^{1/p}=0.$$

A point $x\in\mathbb{R}^d$ is a p-Lebesgue point of f if for all $\mu>0$,

$$\lim_{r\to 0}\sup_{\substack{i\in\mathbb{N}^d,h>0,\\ 2^{i_k}h<r,k=1,\ldots d}} 2^{-\mu|i|}\left(\frac{1}{(2h)^d 2^{|i|}}\int_{-2^{i_1}h}^{2^{i_1}h}\cdots\int_{-2^{i_d}h}^{2^{i_d}h}|f(x-s)-f(x)|^p\,ds\right)^{1/p}=0.$$

Theorem

Almost every point $x \in \mathbb{R}^d$ *is a p-Lebesgue point of* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ *if* $1 \le p < \infty$.

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^d \theta_j \left(\frac{|t_j|}{T_j} \right) \right) \widehat{f}(t) e^{2\pi \imath x \cdot t} \, dt.$$

Theorem (Weisz, 2017)

Let $\theta_i \in L_1(\mathbb{R})$, $1 \le p < \infty$ *and* $1/p + 1/q = 1$ *and* $\mu > 0$. *If* $\widehat{\theta}_i \in E_q^\mu(\mathbb{R})$ $(i = 1, \ldots, d)$, *then*

$$\lim_{T \to \infty, T \in \mathbb{R}_\delta^d} \sigma_T^\theta f(x) = f(x)$$

for all p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$.

Restricted rectangular summability over a cone-like set

Suppose that $\gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and continuous function such that $\lim_\infty \gamma_j = \infty$ and $\lim_{+0} \gamma_j = 0$. Suppose that there exist $c_j, \xi > 1$ such that

$$\gamma_j(\xi x) = c_j \gamma_j(x) \qquad (x > 0, j = 2, \ldots, d,).$$

For convenience we extend the notations for $j = 1$ by $\gamma_1 := \mathcal{I}$ and $c_1 = \xi$. Here $\mathcal{I}$ denotes the identity function $\mathcal{I}(x) = x$. Let

$$\gamma = (\gamma_1, \ldots, \gamma_d) \qquad \text{and} \qquad \delta = (\delta_1, \ldots, \delta_d)$$

with $\delta_1 = 1$ and fixed $\delta_j \geq 1$ $(j = 2, \ldots, d)$.

The cone-like set is defined by

$$\mathbb{R}^d_{\gamma,\delta} := \{T \in \mathbb{R}^d_+ : \delta_j^{-1}\gamma_j(T_1) \le T_j \le \delta_j\gamma_j(T_1), j = 2, \dots, d\}.$$

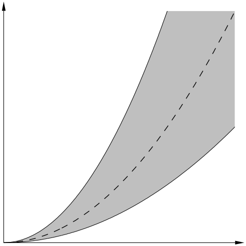

Figure: The cone-like set for $d = 2$.

Recall the definition of the p-Lebesgue points:

$$\lim_{r\to 0} \sup_{\substack{i\in\mathbb{N}^d, h>0,\\ 2^{i_k} h<r, k=1,\ldots d}} 2^{-\mu|i|} \left(\frac{1}{(2h)^d 2^{|i|}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-s)-f(x)|^p \, ds \right)^{1/p} = 0.$$

A point $x \in \mathbb{R}^d$ is a modified p-Lebesgue point of f if for all $\mu > 0$,

$$\lim_{r\to 0} \sup_{\substack{i\in\mathbb{N}^d, h>0,\\ \xi^{i_k} h<r, k=1,\ldots d}} \left(\prod_{j=1}^d \gamma_j(\xi^{i_j})^{-\mu} \right) \left(\frac{1}{\prod_{j=1}^d (2\gamma_j(\xi^{i_j} h))} \int_{-\gamma_1(\xi^{i_1}h)}^{\gamma_1(\xi^{i_1}h)} \cdots \int_{-\gamma_d(\xi^{i_d}h)}^{\gamma_d(\xi^{i_d}h)} |f(x-s)-f(x)|^p \, ds \right)^{1/p} = 0.$$

Theorem

Almost every point $x \in \mathbb{R}^d$ is a modified p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ if $1 \leq p < \infty$.

Theorem (Weisz, 2017)

Let $\theta_i \in L_1(\mathbb{R})$, $1 \leq p < \infty$ and $1/p + 1/q = 1$ and $\mu > 0$. If $\widehat{\theta}_i \in E_q^\mu(\mathbb{R})$ $(i = 1, \ldots, d)$, then

$$\lim_{T \to \infty, T \in \mathbb{R}^d_{\gamma,\delta}} \sigma_T^\theta f(x) = f(x)$$

for all modified p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$.